

**MULTIDIMENSIONAL COCHRAN AND LEE TYPE INEQUALITIES  
 WITH WEIGHTS**

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ABSTRACT. Let  $0 < p \leq q < \infty$ ,  $f$ ,  $V$ , and  $W$  be positive measurable functions on  $\mathbb{R}^n$  and let  $B_x$  be a ball in  $\mathbb{R}^n$  with centre at origin. A general inequality of the type

$$\left( \int_{\mathbb{R}^n} \left[ \exp \frac{1}{|B_x|} \int_{B_x} \ln f(y) dy \right]^q W(x) dx \right)^{1/q} \leq \leq C \left( \int_{\mathbb{R}^n} f^p(x) V(x) dx \right)^{1/p} \quad (1)$$

is proved. Moreover, for the power case  $W(x) = |B_x|^a$ ,  $V(x) = |B_x|^b$  a necessary and sufficient condition for (1) to hold is proved and stable estimates for the least constant in (1) is given. Also the corresponding companion inequalities, where  $\int_{B_x}$  is replaced by  $\int_{\mathbb{R}^n \setminus B_x}$ , are pointed out.

In their paper [2] J. A. Cochran and C. S. Lee proved the inequality

$$\int_0^\infty \left[ \exp \left( \varepsilon x^{-\varepsilon} \int_0^x t^{\varepsilon-1} \ln f(y) dy \right) \right] x^a dx \leq e^{\frac{a+1}{\varepsilon}} \int_0^\infty x^a f(x) dx, \quad (2)$$

where  $a, \varepsilon$  are real numbers with  $\varepsilon > 0$ ,  $f$  is a positive function defined on  $(0, \infty)$  and the constant  $e^{\frac{a+1}{\varepsilon}}$  is the best possible. This inequality, in fact, is

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2000 *Mathematics Subject Classification.* 26D15, 26D07.

*Key words and phrases.* Inequalities, Multidimensional inequalities, geometric mean inequalities, sharp constant.

a generalization of the famous Knopp inequality [11], which is obtained by taking  $\varepsilon = 1$  and  $a = 0$  in (2). Inequalities of the type (2) and its analogues has further been investigated and generalized by many authors e.g., see [4], [5], [6], [7], [8], [9], [10], [12], [13], [14], [15], [16] and [17].

Very recently, A. Čižmešija, J. Pečarić and I. Perić [1, Th 9, formula (23)] proved an  $n$ -dimensional analogue of (2) by replacing the intervals  $(0, \infty)$  by  $\mathbb{R}^n$  and the means are considered over the balls in  $\mathbb{R}^n$  centered at origin. Their inequality reads:

$$\begin{aligned} \int_{\mathbb{R}^n} \left[ \exp \left( \varepsilon |B_x|^{-\varepsilon} \int_{B_x} |B_y|^{\varepsilon-1} \ln f(y) dy \right) \right] |B_x|^a dx &\leq \\ &\leq e^{\frac{a+1}{\varepsilon}} \int_{\mathbb{R}^n} f(x) |B_x|^a dx, \end{aligned} \quad (3)$$

where  $a \in \mathbb{R}$ ,  $\varepsilon > 0$ ,  $f$  is a positive function on  $\mathbb{R}^n$ ,  $B_x$  is a ball in  $\mathbb{R}^n$  with radius  $|x|$ ,  $x \in \mathbb{R}^n$ , centered at the origin and  $|B_x|$  is its volume.

In this paper we further generalize the inequality (3) to a more general situation. First we prove that a more general inequality

$$\left( \int_{\mathbb{R}^n} \left[ \exp \left( \frac{1}{|B_x|} \int_{B_x} \ln f(y) dy \right) \right]^q W(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} f^p(x) V(x) dx \right)^{\frac{1}{p}} \quad (4)$$

holds, where  $f$  is a positive function on  $\mathbb{R}^n$ ,  $W$  and  $V$  are arbitrary weight functions on  $\mathbb{R}^n$  and  $0 < p \leq q < \infty$  (See Theorem 1). In particular, if  $W(x) = |B_x|^a$  and  $V(x) = |B_x|^b$ , then the corresponding Cochran-Lee type inequality reads

$$\begin{aligned} \left( \int_{\mathbb{R}^n} \left[ \exp \left( \varepsilon |B_x|^{-\varepsilon} \int_{B_x} |B_x|^{\varepsilon-1} \ln f(y) dy \right) \right]^q |B_x|^a dx \right)^{1/q} &\leq \\ &\leq C \left( \int_{\mathbb{R}^n} f^p(x) |B_x|^b dx \right), \end{aligned} \quad (5)$$

where

$$C = \left( \frac{p}{q} \right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p} - \frac{1}{q}} e^{\frac{b+1}{\varepsilon p}} \quad \text{and} \quad a = \frac{q}{p}(b+1) - 1.$$

Note that for the case  $q = p$  the constant in (5) is sharp and coincides with the (Čižmešija-Pečarić-Perić)inequality (3) when  $p = q = 1$  and (5) does not in general hold if  $a \neq \frac{q}{p}(b+1) - 1$ . The last statement follows from our Theorem 2, where necessary and sufficient conditions for the inequality (5) to hold is derived and where also stable upper and lower estimates of the least constant  $C$  (the operator norm) are pointed out. Further, we also study the inequality which is a companion to that of (4) in which the integral  $\int_{\mathbb{R}^n}$  is replaced by  $\int_{B_x}$  (see Theorem 3) and as a consequence of

this result we obtain an inequality like that in (5) also for this case (see Theorem 4) thereby generalizing also the companion inequality of (3) (see [1, Th 9, formula (24)]).

Before we go further, we give some notation and preliminaries which we shall use in the sequel. By weights  $W, V$ , we shall mean positive locally integrable functions on  $\mathbb{R}^n$ . The symbol  $B_x$  denotes a ball in  $\mathbb{R}^n$  with radius  $|x|$ ,  $x \in \mathbb{R}^n$  and centered at origin. The surface of  $B_1$  (unit ball in  $\mathbb{R}^n$ ) is denoted by  $S_{n-1}$  and its area by  $|S_{n-1}|$ . The volume of  $B_x$ , denoted by  $|B_x|$ , can be calculated as

$$|B_x| = \int_{|y| < |x|} dy = \int_{S_{n-1}} \left( \int_0^{|x|} |x|t^{n-1} dt \right) d\sigma = \frac{|x|^n |S_{n-1}|}{n}. \quad (6)$$

We also need the following  $n$ -dimensional Hardy inequality proved by P. Drábek, H. Heinig and H. Kufner [3]:

**Theorem A.** *Let  $W$  and  $V$  be weight functions on  $\mathbb{R}^n$  and  $1 < p \leq q < \infty$ . Then the inequality*

$$\left( \int_{\mathbb{R}^n} \left( \int_{B_x} f(y) dy \right)^q W(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} f^p(x) V(x) dx \right)^{\frac{1}{p}} \quad (7)$$

holds for all non-negative functions  $f$  if and only if

$$\mathcal{A}_n := \sup_{x \in \mathbb{R}^n} \left( \int_{\mathbb{R}^n \setminus B_x} W(y) dy \right)^{\frac{1}{q}} \left( \int_{B_x} V^{\frac{1}{1-p}}(y) dy \right)^{\frac{p-1}{p}} < \infty$$

and the best possible constant  $C$  in (7) can be estimated in the following way :

$$\mathcal{A}_n \leq C \leq \mathcal{A}_n \left( \frac{p}{p-1} \right)^{\frac{p-1}{p}} p^{\frac{1}{q}}.$$

Our first main result is given as follows :

**Theorem 1.** *Let  $0 < p \leq q < \infty$  it and  $W, V$  be weight functions on  $\mathbb{R}^n$ . If*

$$D_n := \sup_{x \in \mathbb{R}^n} |B_x|^{\frac{1}{q} - \frac{1}{p}} W_1^{\frac{1}{q}}(x) \left[ \exp \left( \frac{1}{|B_x|} \int_{B_x} \ln \frac{1}{V(y)} dy \right) \right]^{\frac{1}{p}} < \infty, \quad (8)$$

where  $W_1$  is the arithmetic average, i.e.,

$$W_1(x) = \frac{1}{|S_{n-1}|} \int_{S_{n-1}} W(|x|\sigma) d\sigma,$$

then

$$\left( \int_{\mathbb{R}^n} \left[ \exp \left( \frac{1}{|B_x|} \int_{B_x} \ln f(y) dy \right) \right]^q W(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} f^p(x) V(x) dx \right)^{\frac{1}{p}} \quad (9)$$

holds for all positive functions with some finite constant  $C$ . Moreover, the least possible constant such that (9) holds can be estimated as follows:

$$C \leq \left( \frac{p}{q} \right)^{1/q} e^{1/p} D_n. \quad (10)$$

*Proof.* By applying Theorem A, with  $\frac{p}{\alpha}$ ,  $\frac{q}{\alpha}$ ,  $W(x)|B_x|^{-\frac{q}{\alpha}}$ ,  $0 < \alpha < p$ , in place of  $p$ ,  $q$ ,  $W(x)$  and replacing  $f$  by  $f^\alpha$ , we get that if

$$\mathcal{A}_{n,\alpha} := \sup_{x \in \mathbb{R}^n} \left( \int_{\mathbb{R}^n \setminus B_x} W(y) |B_y|^{-\frac{q}{\alpha}} dy \right)^{\frac{1}{q}} \left( \int_{B_x} V^{\frac{\alpha}{\alpha-p}}(y) dy \right)^{\frac{p-\alpha}{\alpha p}} < \infty,$$

then the inequality

$$\left( \int_{\mathbb{R}^n} \left( \frac{1}{|B_x|} \int_{B_x} f^\alpha(y) dy \right)^{\frac{q}{\alpha}} W(x) dx \right)^{\frac{1}{q}} \leq C_\alpha \left( \int_{\mathbb{R}^n} f^p(x) V(x) dx \right)^{\frac{1}{p}}$$

holds for all  $f > 0$  and the constant  $C_\alpha$  satisfies

$$C_\alpha \leq \mathcal{A}_{n,\alpha} \left( \frac{p}{p-\alpha} \right)^{\frac{p-\alpha}{\alpha p}} \cdot \left( \frac{p}{\alpha} \right)^{\frac{1}{q}}.$$

We note that

$$\mathcal{A}_{n,\alpha} \left( \frac{p}{p-\alpha} \right)^{\frac{p-\alpha}{\alpha p}} \cdot \left( \frac{p}{\alpha} \right)^{\frac{1}{q}} = A_{n,\alpha} \left( \frac{p}{p-\alpha} \right)^{\frac{p-\alpha}{\alpha p}} p^{\frac{1}{q}},$$

where

$$\begin{aligned} A_{n,\alpha} &:= \sup_{x \in \mathbb{R}^n} \left( \frac{1}{\alpha} \right)^{\frac{1}{q}} \times \\ &\times \left( \int_{\mathbb{R}^n \setminus B_x} W(y) |B_y|^{-\frac{q}{\alpha}} dy \right)^{\frac{1}{q}} \left( \int_{B_x} V^{\frac{\alpha}{\alpha-p}}(y) dy \right)^{\frac{p-\alpha}{\alpha p}} < \infty. \end{aligned} \quad (11)$$

We also note that the scale of power means

$$\left( \frac{1}{|B_x|} \int_{B_x} f^\alpha(y) dy \right)^{1/\alpha}$$

is increasing in  $\alpha$  and converges to the geometric mean

$$\exp \left( \frac{1}{|B_x|} \int_{B_x} \ln f(y) dy \right)$$

as  $\alpha \rightarrow 0$  and  $\lim_{\alpha \rightarrow 0^+} \left(\frac{p}{p-\alpha}\right)^{\frac{p-\alpha}{\alpha p}} = e^{1/p}$ . Thus the inequality (9) holds with

$$C \leq A_n p^{\frac{1}{q}} e^{\frac{1}{p}}, \quad (12)$$

where

$$A_n := \lim_{\alpha \rightarrow 0^+} A_{n,\alpha}.$$

Next we note that

$$\begin{aligned} A_{n,\alpha} &= \sup_{x \in \mathbb{R}^n} \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} |x|^{\frac{n}{q} - \frac{n}{p}} \left(\frac{|S_{n-1}|}{n}\right)^{-\frac{1}{p}} \left( \int_{\mathbb{R}^n \setminus B_x} W(y) \left(\frac{|xt|}{|y|}\right)^{\frac{nq}{\alpha}} |x|^{-n} dy \right)^{\frac{1}{q}} \times \\ &\quad \times \left( \frac{1}{|B_x|} \int_{B_x} V^{\frac{\alpha}{\alpha-p}}(y) dy \right)^{\frac{p-\alpha}{\alpha p}}. \end{aligned}$$

Now using the variable transformation  $y = |x|z$ , in the first integral we obtain

$$\begin{aligned} I_\alpha &:= \frac{1}{\alpha} \int_{\mathbb{R}^n \setminus B_x} W(y) \left(\frac{|x|}{|y|}\right)^{\frac{nq}{\alpha}} |x|^{-n} dy = \frac{1}{\alpha} \int_{\mathbb{R}^n \setminus B_1} W(|x|z) |z|^{-\frac{nq}{\alpha}} dz = \\ &= \frac{1}{\alpha} \int_{S_{n-1}} \int_1^\infty W(|x|t\sigma) t^{n - \frac{nq}{\alpha} - 1} dt d\sigma. \end{aligned} \quad (13)$$

Moreover, we note that

$$\chi_{[1,\infty)}(t) n \left(\frac{q}{\alpha} - 1\right) t^{n - \frac{nq}{\alpha} - 1} \rightarrow \delta_1(t) \text{ as } \alpha \rightarrow 0^+,$$

where  $\delta_1(t)$  is the Dirac delta function at  $t = 1$  (point unit mass at  $t = 1$ ). This implies that

$$I_\alpha \rightarrow \frac{1}{nq} \int_{S_{n-1}} W(|x|\sigma) d\sigma = \frac{1}{nq} |S_{n-1}| W_1(x) \text{ as } \alpha \rightarrow 0^+. \quad (14)$$

Now, by taking the limit  $\alpha \rightarrow 0^+$  in (11) and using (6), (13) and (14) we find that

$$\begin{aligned} A_n &= \sup_{x \in \mathbb{R}^n} q^{-\frac{1}{q}} |x|^{\frac{n}{q} - \frac{n}{p}} \left(\frac{|S_{n-1}|}{n}\right)^{\frac{1}{q} - \frac{1}{p}} W_1^{\frac{1}{q}}(x) \left[ \exp \left( \frac{1}{|B_x|} \int_{B_x} \ln \frac{1}{V(y)} dy \right) \right]^{\frac{1}{p}} = \\ &= q^{-\frac{1}{q}} D_n. \end{aligned} \quad (15)$$

By using (8) combined with (12) and (15) we obtain (9), (10) and the proof is complete.

Next we will prove that the following stronger statement holds for the special case  $V(x) = |B_x|^b$  and  $W(x) = |B_x|^a$ :

**Theorem 2.** Let  $0 < p \leq q < \infty$ ,  $a, b \in \mathbb{R}$  and  $\varepsilon \in \mathbb{R}_+$ . Then

$$\begin{aligned} \left( \int_{\mathbb{R}^n} \left[ \exp \left( \varepsilon |B_x|^{-\varepsilon} \int_{B_x} |B_y|^{\varepsilon-1} \ln f(y) dy \right) \right]^q |B_x|^a dx \right)^{\frac{1}{q}} &\leq \\ &\leq C \left( \int_{\mathbb{R}^n} f^p(x) |B_x|^b dx \right)^{\frac{1}{p}} \end{aligned} \quad (16)$$

holds for all positive functions  $f$  for some finite constant  $C$  if and only if

$$\frac{a+1}{q} = \frac{b+1}{p} \quad (17)$$

and the best constant  $C$  in (16) satisfies

$$\left( \frac{p}{q} \right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{\frac{b+1}{\varepsilon p}-\frac{1}{p}} \leq C \leq \left( \frac{p}{q} \right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{\frac{b+1}{\varepsilon p}}.$$

*Remark 1:* We note that by applying Theorem 2 with  $p = q = 1$  we obtain a more general statement than the (Čižmešija-Pečarić-Perić) estimate (3).

*Proof.* Assume that (17) holds and let  $f$  be an arbitrary positive function on  $\mathbb{R}^n$ . By writing (16) in polar coordinates we find that

$$\begin{aligned} &\left( \int_{S_{n-1}} \int_0^\infty x^{na+n-1} \left[ \exp \left( \frac{\varepsilon n^\varepsilon}{x^{n\varepsilon} |S_{n-1}|^\varepsilon} \int_{S_{n-1}} \int_0^{|x|} \left( \frac{|S_{n-1}|}{n} \right)^{\varepsilon-1} y^{n\varepsilon-1} \times \right. \right. \\ &\times \left. \left. \ln f(y\sigma) dy d\sigma \right) \right]^q dx d\tau \right)^{1/q} \leq \\ &\leq C \left( \int_{S_{n-1}} \int_0^\infty x^{nb+n-1} \left( \frac{|S_{n-1}|}{n} \right)^b f^p(x\tau) dx d\tau \right)^{1/p}. \end{aligned}$$

By first making the variable transformation  $y = r^{\frac{1}{\varepsilon}}$  and after that the transformation  $x = t^\varepsilon$  we find that this inequality can be rewritten as

$$\begin{aligned} &\left( \int_{S_{n-1}} \int_0^\infty \left( \frac{|S_{n-1}|}{n} \right)^a t^{n(\frac{a+1}{\varepsilon}-1)} t^{n-1} \exp \left( \frac{nq}{|S_{n-1}| t^n} \int_{S_{n-1}} \int_0^{|t|} r^{n-1} \times \right. \right. \\ &\times \left. \left. \ln f(r^{\frac{1}{\varepsilon}} \sigma) dr d\sigma \right) \frac{1}{\varepsilon} dt d\tau \right)^{\frac{1}{q}} \leq \\ &\leq C \left( \int_{S_{n-1}} \int_0^\infty \left( \frac{|S_{n-1}|}{n} \right)^b t^{n(\frac{b+1}{\varepsilon}-1)} t^{n-1} f^p(t^{\frac{1}{\varepsilon}} \tau) \frac{1}{\varepsilon} dt d\tau \right)^{\frac{1}{p}}, \end{aligned}$$

i.e., in view of (6) and the assumption (17)

$$\begin{aligned} & \left( \int_{\mathbb{R}^n} |B_t|^{\left(\frac{a+1}{\varepsilon}-1\right)} \exp\left(\frac{q}{|B_t|} \int_{B_t} \ln f_1(s) ds\right) dt \right)^{\frac{1}{q}} \leq \\ & \leq C \left( \frac{|S_{n-1}|}{n} \right)^{\left(\frac{b+1}{p}-\frac{a+1}{q}\right)\left(1-\frac{1}{\varepsilon}\right)} \varepsilon^{\frac{1}{q}-\frac{1}{p}} \left( \int_{\mathbb{R}^n} |B_t|^{\left(\frac{b+1}{\varepsilon}-1\right)} f_1^p(t) dt \right)^{\frac{1}{p}}, \end{aligned} \quad (18)$$

where  $f_1(r\sigma) = f(r^{\frac{1}{\varepsilon}}\sigma)$ . We will now apply Theorem 1 with the weights  $W(x) = |B_x|^{\left(\frac{a+1}{\varepsilon}-1\right)}$  and  $V(x) = |B_x|^{\left(\frac{b+1}{\varepsilon}-1\right)}$  and note that in this case

$$D_n = \sup_{x \in \mathbb{R}^n} |B_x|^{\left(\frac{a+1}{q\varepsilon}-\frac{1}{p}\right)} \left( \exp \frac{1}{|B_x|} \int_{B_x} \ln |B_y|^{-\frac{b+1}{\varepsilon}+1} dy \right)^{\frac{1}{p}}. \quad (19)$$

Moreover, some straightforward calculations show that

$$\begin{aligned} \int_{B_x} \ln |B_y|^{-\frac{b+1}{\varepsilon}+1} dy &= \int_{S_{n-1}} \int_0^{|x|} y^{n-1} \left(1 - \frac{b+1}{\varepsilon}\right) \ln \left(\frac{|S_{n-1}|}{n} y^n\right) dy d\sigma = \dots \\ &= |B_x| \left( \ln |B_x|^{-\frac{b+1}{\varepsilon}+1} + \frac{b+1}{\varepsilon} - 1 \right). \end{aligned} \quad (20)$$

Thus, by combining (19) and (20) we find that

$$D_n = e^{\frac{b+1}{\varepsilon p}-\frac{1}{p}} \sup_{x \in \mathbb{R}^n} |B_x|^{\left(\frac{a+1}{q}-\frac{b+1}{p}\right)\frac{1}{\varepsilon}}$$

so that, according to the assumption (17)

$$D_n = e^{\frac{b+1}{\varepsilon p}-\frac{1}{p}} < \infty$$

which, by Theorem 1, means that (16) holds for each positive function  $f$  and, moreover, by (10) and (18),

$$C \leq \left(\frac{p}{q}\right)^{1/q} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{\frac{b+1}{\varepsilon p}}.$$

On the contrary, assume that (16) holds for all positive functions  $f$ . Again we rewrite (16) on the form (18) and apply this inequality with the test function

$$f_x(t) = |B_t|^{\frac{1}{p}\left(1-\frac{b+1}{\varepsilon}\right)} \chi_{B_x},$$

where  $x \in \mathbb{R}^n$  is arbitrary. We obtain

$$\left( \int_{B_x} |B_t|^{\frac{a+1}{\varepsilon}-1} \left( \exp \frac{q}{|B_t|} \int_{B_t} \ln |B_s|^{\frac{1}{p}\left(1-\frac{b+1}{\varepsilon}\right)} ds \right) dt \right)^{1/q} \leq C_0 |B_x|^{1/p}, \quad (21)$$

where

$$C_0 = \left( \frac{|S_{n-1}|}{n} \right)^{\left( \frac{b+1}{p} - \frac{a+1}{q} \right) \left( 1 - \frac{1}{\varepsilon} \right)} \varepsilon^{\frac{1}{q} - \frac{1}{p}} C. \quad (22)$$

By now again using polar coordinates and making some straightforward calculations similar to those before we find that

$$\int_{B_t} \ln |B_s|^{\frac{1}{p} \left( 1 - \frac{b+1}{\varepsilon} \right)} ds = \dots = |B_t| \ln |B_t|^{\frac{1}{p} \left( 1 - \frac{b+1}{\varepsilon} \right)} + \frac{1}{p} \left( \frac{b+1}{\varepsilon} - 1 \right).$$

Hence, in view of (21)

$$e^{\frac{b+1}{\varepsilon p} - \frac{1}{p}} \left( \int_{B_x} |B_t|^{\frac{a+1}{\varepsilon} - 1 + \frac{a}{p} \left( 1 - \frac{b+1}{\varepsilon} \right)} dt \right)^{1/q} \leq C_0 |B_x|^{1/p}$$

which we rewrite as

$$\left( \int_{B_x} |B_t|^{\frac{a}{p_0} - 1} dt \right)^{1/q} \leq e^{\frac{1}{p} - \frac{b+1}{\varepsilon p}} C_0 |B_x|^{1/p}, \quad (23)$$

where

$$p_0 = p/(\alpha + 1), \alpha = \frac{a+1}{\varepsilon} \frac{p}{q} - \frac{b+1}{\varepsilon}. \quad (24)$$

By using polar coordinates in (23) we find that

$$\left( \int_{B_x} |B_t|^{\frac{a}{p_0} - 1} dt \right)^{1/q} = |B_x|^{1/p_0} \left( \frac{p_0}{q} \right)^{1/q} \leq e^{\frac{1}{p} - \frac{b+1}{\varepsilon p}} C_0 |B_x|^{1/p},$$

i.e., that

$$|B_x|^{\frac{1}{p_0} - \frac{1}{p}} \leq \left( \frac{q}{p_0} \right)^{1/q} e^{\frac{1}{p} - \frac{b+1}{\varepsilon p}} C_0. \quad (25)$$

This can be true for each  $\bar{x} \in \mathbb{R}^n$  if and only if  $p_0 = p$  which means that  $\alpha = 0$  so that, by (24), (17) holds. Moreover, according to (22) and (25) we see that

$$C \geq \left( \frac{p}{q} \right)^{1/q} e^{\frac{b+1}{\varepsilon p} - \frac{1}{p} \frac{1}{\varepsilon} - \frac{1}{q}},$$

and the proof is complete.  $\square$

*Remark 2:* If we take  $p = q$  so that  $a = b$ , then the inequality (16) holds with the constant  $C = e^{\frac{b+1}{\varepsilon p}}$ . By using the functions

$$f_\delta(x) = \begin{cases} e^{-\frac{b+1}{\varepsilon p}} |B_1|^{-(b+1)} |x|^{-\frac{a}{p}(b+1-\varepsilon\delta)} & , x \in B_1, \\ e^{-\frac{b+1}{\varepsilon p}} |B_1|^{-(b+1)} |x|^{-\frac{a}{p}(b+1+\varepsilon\delta)} & , x \in \mathbb{R}^n \setminus B_1, \end{cases}$$

in (16) and letting  $\delta \rightarrow 0^+$ , it can be shown that the constant  $C = e^{\frac{b+1}{\varepsilon p}}$  is sharp.

We also state the following complement of Theorem 1:

**Theorem 3.** *Let  $0 < p \leq q < \infty$  and  $W, V$  be weight functions defined on  $\mathbb{R}^n$ . If*

$$\tilde{D}_n := \sup_{x \in \mathbb{R}^n} |B_x|^{\frac{1}{q} - \frac{1}{p}} \tilde{W}^{\frac{1}{q}}(x) \left[ \exp \left( \frac{1}{|B_x|} \int_{B_x} \ln \frac{1}{\tilde{V}(y)} dy \right) \right]^{\frac{1}{p}} < \infty,$$

where

$$\tilde{W}(s) = W_1 \left( s^{-\frac{1}{\varepsilon}} \right) \frac{1}{\varepsilon} |s|^{-n(1+\frac{1}{\varepsilon})}, \quad \tilde{V}(s) = V \left( s^{-\frac{1}{\varepsilon}} \right) \frac{1}{\varepsilon} |s|^{-n(1+\frac{1}{\varepsilon})},$$

then

$$\begin{aligned} \left( \int_{\mathbb{R}^n} \left[ \exp \left( \varepsilon |B_x|^\varepsilon \int_{\mathbb{R}^n \setminus B_x} |B_y|^{-\varepsilon-1} \ln f(y) dy \right) \right]^q W(x) dx \right)^{\frac{1}{q}} &\leq \\ &\leq C \left( \int_{\mathbb{R}^n} f^p(x) V(x) dx \right)^{\frac{1}{p}} \end{aligned} \quad (26)$$

holds for all positive functions  $f$ . Moreover, the least possible constant such that (26) holds can be estimated as follows:

$$C \leq \left( \frac{p}{q} \right)^{\frac{1}{q}} e^{\frac{1}{p}} \tilde{D}_n.$$

*Proof.* By using polar coordinates, (6) and some variable transformations similar to (but not the same) as those used in the proof of Theorem 2, the equation (26) can be transformed to

$$\begin{aligned} \left( \int_{\mathbb{R}^n} \left[ \exp \left( \frac{1}{|B_s|} \int_{B_s} \ln f(z^{-\frac{1}{\varepsilon}}) dz \right) \right]^q \tilde{W}(s) ds \right)^{\frac{1}{q}} &\leq \\ &\leq C \left( \int_{\mathbb{R}^n} f^p \left( s^{-\frac{1}{\varepsilon}} \right) \tilde{V}(s) ds \right)^{\frac{1}{p}}. \end{aligned}$$

Now by applying Theorem 1 and again using polar coordinates, (6) and some variable transformations again, we get the result.  $\square$

For the power weight case  $W(x) = |B_x|^a$ ,  $V(x) = |B_x|^b$ , we have the following more precise result:

**Theorem 4.** *Let  $0 < p \leq q < \infty$ ,  $\varepsilon > 0$  and  $a, b \in \mathbb{R}$ . The inequality*

$$\left( \int_{\mathbb{R}^n} \left[ \exp(\varepsilon |B_x|^\varepsilon \int_{\mathbb{R}^n \setminus B_x} |B_y|^{-\varepsilon-1} \ln f(y) dy) \right]^q |B_x|^a dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} f^p(x) |B_x|^b dx \right)^{\frac{1}{p}} \quad (27)$$

holds for all positive functions  $f$  on  $\mathbb{R}^n$  and some finite constant  $C$ , if and only if

$$\frac{a+1}{q} = \frac{b+1}{p}.$$

Moreover, the least possible constant  $C$  such that (27) holds can be estimated as follows:

$$\left(\frac{p}{q}\right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{-\left(\frac{b+1}{\varepsilon p} + \frac{1}{p}\right)} \leq C \leq \left(\frac{p}{q}\right)^{\frac{1}{q}} \varepsilon^{\frac{1}{p}-\frac{1}{q}} e^{-\frac{b+1}{\varepsilon p}}.$$

*Remark 3:* If we take  $p = q$  so that  $a = b$ , then the inequality (27) holds with the constant  $C = e^{-\frac{b+1}{\varepsilon p}}$ . By using the functions

$$f_\delta(x) = \begin{cases} e^{\frac{b+1}{\varepsilon p} |B_1|^{-(b+1)} |x|^{-\frac{\alpha}{p}(b+1-\varepsilon\delta)}} & , x \in B_1, \\ e^{\frac{b+1}{\varepsilon p} |B_1|^{-(b+1)} |x|^{-\frac{\alpha}{p}(b+1+\varepsilon\delta)}} & , x \in \mathbb{R}^n \setminus B_1, \end{cases}$$

in (27) and letting  $\delta \rightarrow 0^+$ , it can be shown that the constant  $C = e^{-\frac{b+1}{\varepsilon p}}$  is sharp. In particular, by combining this fact with our previous Remark 2 we see that Theorems 2 and 4 with  $p = q = 1$  give a more general information than that in [1, Theorem 9].

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(Received 10.05.2002.)

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