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ON ONE MODIFIED VERSION OF MFS FOR CONFORMAL MAPPING OF INFINITE DOMAIN ONTO THE UNIT DISK

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Abstract. The method for approximate construction of a function conformally mapping an infinite domain onto the unit disk is given. It is applied to the modified version of the method of fundamental solutions. Examples are given to illustrate effectiveness, simplicity and high accuracy of the method.

1. Introduction

As is known, the basic problem of the theory of conformal mappings consists in the following [1]–[3]: given two domains (of the same connectivity) D and G, we have to construct analytic (holomorphic) function $\zeta = f(z)$ which maps conformally the domain D onto the domain G, and vice versa. As for the existence of the function of conformal mapping $\zeta = f(z)$, the question is solved positively by the Riemann theorem (see [1]–[3]). However, exact or approximate construction of this function for the given domains is a difficult mathematical problem, though its solution is very important from applicational point of view. In particular, application of the method of conformal mapping to the solution of practical problems (such as hydro– and aerodynamics, elasticity, filtration, the theory of electric, magnetic and thermal fields, etc.) is connected with the development of methods for constructing conformally mapping functions. As far as conformally mapping functions can be written explicitly only for a rather narrow class of domains, one has to resort to approximate methods.

Elaboration of approximate methods of conformal mapping is carrying out for almost a century and the literature devoted to these problems in-

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volves an overwhelming number of papers. Monographs [2,4] deal just with this question. Their authors present the most fundamental works in this area. Despite the great attention, the problem of constructing a simple, but rather effective and exact method of mapping is far from being solved completely. Consequently, a number of scientific centers are engaged in constructing conformal mapping by means of up-to-date electronic computers.

Below we suggest one scheme for an approximate construction of the function mapping conformally the given infinite domain onto the unit disk.

2. Modified version of the method of fundamental solutions

Let D be an infinite part of the plane $z = x + iy \equiv (x, y)$ bounded by one closed Jordan contour $S\big(S = \sum\limits_{j=1}^{l} S^j\big)$. Assume also that the parametric equations of smooth curves $S^j : x = x^j(t)$, $y = y^j(t)$, where $\alpha_j \le t \le \beta_j$, are known.

We consider the following Dirichlet exterior boundary value problem for the Laplace equation. The problem is stated as follows: find a function $u(x, y) \in C^2(D) \cap C(\overline{D})$, satisfying the conditions

$$\Delta u(z) = o, \quad \forall z \equiv (x, y) \in D, \tag{2.1}$$

$$u(z) = g(z), \quad \forall z \in S, \tag{2.2}$$

$$u(z) = c + O\left(\frac{1}{|z|}\right) \quad \text{for} \quad |z| \to \infty, \tag{2.3}$$

where $g(z) \equiv g(x, y)$ is a given continuous function on the contour S and c is a real constant, provided $|c| < \infty$. Problems of type (2.1), (2.2), (2.3) arise in hydromechnics, theory of elasticity and in some other areas of mathematical physics. It is known [5], [6] that problem (2.1), (2.2), (2.3) is correct, i.e., a solution exists, it is unique and depends continuously on the initial data. The request for the solution to be bounded at infinity is very essential for its uniqueness [5], [6]. It can be easily shown that if we fix in advance the constant c, this will be a rather strong restriction. Indeed, since the minimax principle [5], [6] for the function $u(x, y)$ is fulfilled under conditions (2.1), (2.2), (2.3), problem (2.1), (2.2), (2.3) with hitherto fixed c may turn out to be unsolvable in general. To avoid this fact the constant c should be defined from the boundary condition (2.2) when solving problem (2.1), (2.2), (2.3).

To solve approximately the internal and external boundary value problems [7,8], one can use the method of Kupradze-Aleksidze (MFS). In our case, despite the fact that the system

$$\{\varphi_k(z)\}_{k=1}^{\infty} = \{\ln|z - \widetilde{z}_k|\}_{k=1}^{\infty}, \quad z \in S, \tag{2.4}$$

of fundamental solutions of the Laplace operator ($\{\widetilde{z}_k\}_{k=1}^{\infty}$ is a countable set of points lying densely everywhere on an auxiliary closed Ljapunov contour

$\widetilde{S}$ which lies inside the finite domain B bounded by the contour S and min $\rho(S, \widetilde{S}) > 0$, where ρ is the distance between S and $\widetilde{S}$) is linearly independent and complete not only in the space $L_2(S)$ but also in $C(S)$ (see [7], [8]), it is inconvenient for an approximate solution of problem (2.1), (2.2), (2.3). Indeed, when using the MFS, the approximate solution is sought in the form

$$u_N(z) = \sum_{k=1}^{N} a_k^{(N)} \ln |z - \widetilde{z}_k|, \quad z \in \overline{D}, \tag{2.5}$$

where the points $\widetilde{z}_k$ $(k = 1, 2, \dots, N)$ lie "uniformly" on the auxiliary contour $\widetilde{S}$, and $a_k^{(N)}$ are the series expansion of coefficients of the boundary function $g(z)$ with respect to the first N functions of system (2.4). It is obvious that $\lim u_N(z) = \infty$ as $|z| \to \infty$, which implies that condition (2.3) is not fulfilled for the solution to be unique.

Remark 1. In the sequel (see [9]), in solving approximately the boundary value problem, under the contour $\widetilde{S}$ will be meant the Jordan contour representing the boundary of the plane figure $\widetilde{B}(\widetilde{B} \subset B)$. The figure $\widetilde{B}$ is similar to B, oriented in one and the same direction and have one common "center" of gravity. As for the values $\rho(S, \widetilde{S})$ and N, they can be chosen in the process of numerical realization of the algorithm with regard for a posteriori estimates of accuracy of the results.

Remark 2. If we seek for a solution of problem (2.1), (2.2), (2.3) in the form

$$u_N(z) = \sum_{k=1}^{N} a_k^{(N)} ln|z - \widetilde{z}_k| + c_N,$$

under the condition $\sum\limits_{k=1}^{N} a_k^{(N)} = 0$, where c_N is a real constant and $|c_N| < \infty$, it is not difficult to prove that $u_N(\infty) = c_N$. However, while finding the constants $a_k^{(N)}$ $(k = 1, 2, \dots, N)$ and c_N there arise considerable difficulties connected with the investigation of the solvability of the obtained systems and conditionality of their matrices.

To avoid these difficulties, a modified system of fundamental solutions of the Laplace operator

$$\big\{\psi_k(z)\big\}_{k=1}^{\infty} \equiv \big\{\psi_k(z, \widetilde{z}_k\big\}_{k=1}^{\infty} = \left\{ \ln \left| \frac{\widetilde{z}_k - z}{(z - z_0)(\widetilde{z}_k - z_0)} \right| \right\}_{k=1}^{\infty}, \tag{2.6}$$

has been constructed for an approximate solution of problem (2.1), (2.2), (2.3), where z_0 is the "centre" of the domain B and $\{\widetilde{z}_k\}_{k=1}^{\infty}$ is a countable set of points, dense everywhere on the auxiliary Liapunov contour $\widetilde{S}$. The following theorem is valid.

Theorem 1. *The system of functions* $\{\psi_k(z)\}_{k=1}^{\infty}$ *is linearly independent and complete not only in the space* $L_2(S)$*, but also in* $C(S)$*.*

On the basis of the above-stated theorem an approximate solution $u_N(z)$ for problem (2.1), (2.2), (2.3) is sought in the form

$$u_N(z) \equiv u_N(x,y) = \sum_{k=1}^{N} a_k^{(N)} \ln\left|\frac{\widetilde{z}_k - z}{(z-z_0)(\widetilde{z}_k - z_0)}\right|, \quad z \in \overline{D}, \quad (2.7)$$

where auxiliary points (simulation sources) $\widetilde{z}_k$ $(k = 1,2,\dots,N)$ lie "uniformly" on the Jordan contour $\widetilde{S}$, and z_0 is the "centre" of the domains B and $\widetilde{B}$ (see Remark 2). As for the coefficients $a_k^{(N)}$, they can be found (see [7], [8], [9]) from the systems of linear algebraic equations of the form

$$\sum_{k=1}^{N} a_k^{(N)}(\psi_k,\psi_j) = (g,\psi_j), \quad (j = 1,2,\dots,N), \quad (2.8)$$

where $\psi_k(z) = \psi(z,\widetilde{z}_k)$ and $(\psi_k,\psi_j) = \int\limits_S \psi_k(z)\psi_j(z)d_zS$, or by

$$\sum_{k=1}^{N} a_k^{(N)}\psi(z_j,\widetilde{z}_k) = g(z_j), \quad (2.9)$$

where the collocation points z_j $(j = 1,2,\dots,N)$ lie "uniformly" on S. From (2.7), for the approximate value of the constant c we have

$$c_N = \lim_{|z|\to\infty} u_N(z) \equiv u_N(\infty) = \\ = -\sum_{k=1}^{N} a_k^{(N)} \ln|\widetilde{z}_k - z_0| \quad \text{or} \quad |c_N| < \infty. \quad (2.10)$$

It can be easily verified that when solving problem (2.1), (2.2), (2.3) with the use of system (2.6) the following theorem is valid:

Theorem 2. *The sequence of harmonic functions $\{u_N(z)\}$ not only converges uniformly on $\overline{D}$ to $u(z)$, but also the sequence of derivatives of $u_N(z)$ converges to the derivative of $u(z)$ uniformly in D, i.e.,*

$$\lim_{N\to\infty} u_N(z) = u(z), \quad z \in \overline{D}, \quad (2.11)$$
$$\lim_{N\to\infty} \frac{\partial u_N^n(x,y)}{\partial x^m \partial y^{n-m}} = \frac{\partial u^n(x,y)}{\partial x^m \partial y^{n-m}}, \quad (x,y) \in D,$$

where n is a fixed natural number, m is an integer and $n \ge m \ge 0$.

As for the error asymptotics of the solution of problem (2.1), (2.2), (2.3) by system (2.6), there takes place the theorem, which is analogous to the theorem proven in [11]. Namely, the following theorem is valid.

Theorem 3. *If D is an infinite domain bounded by one analytic Jordan contour S, $g(z)$ is an analytic boundary function on S and N is sufficiently large, then there exists a structure of auxiliary points $\widetilde{z}_k$ and collocation points z_k $(k = 1, 2, \dots, N)$, and the constants $c > 0$, N_0, $0 < \tau < 1$, such that*

$$\max_{z \in S} |u_N(z) - g(z)| \leq c\tau^N$$

for all $N \geq N_0$.

This theorem is of theoretical rather than of practical importance, because to define the location of the points $\widetilde{z}_k$ and z_k $(k = 1, 2, \dots, N)$, it is necessary to have a function which maps conformally the unit disk onto the domain D (see [11]). However, the problem of conformal mapping (except for a rather narrow family of domains) is firstly difficult by itself and, secondly, if the contour S has a geometry such that a strong compression or extension takes place when the unit disk is mapped onto D, then the number N_0 is too large (for example, $N_0 > 10000$), which makes difficult the solution of system (2.8),(2.9) (see [12]).

3. Approximate construction of a function conformally mapping an infinite domain onto a unit disk

Let in the plane $z = x + iy$ be given an infinite domain D bounded by one closed Jordan contour S (see Section 2) and in the plane $\zeta = \xi + i\eta$ a unit disk $G(|\zeta| < 1)$ with the boundary γ.

To find the form of the conformally mapping function $\zeta = f(z)$ we pass from the infinite domain D to the finite domain D^* with the boundary S^* using the conformal mapping

$$z^* = z_o + \frac{1}{z - z_o}, \tag{3.1}$$

where z_o is the "centre" of the finite domain B bounded by the contour S. Evidently, for mapping (3.1), the exterior of the disk $C(z_o; 1)$ transforms onto the disk $C(z_o; 1)$, and vice versa.

The function $\zeta = f^*(z^*)$, mapping conformally the domain D^* onto the unit disk G has the form [1]–[2]

$$f^*(z^*) = (z^* - z_0) \exp\big[u^*(z^*) + iv^*(z^*)\big], \tag{3.2}$$

where $u^*(z^*) \equiv u^*(x^*, y^*)$ is a solution of the Dirichlet boundary value problem for the domain D^*,

$$\Delta u^*(z^*) = 0, \quad \forall z^* \in D^*, \tag{3.3}$$

$$u^*(z^*) = -\ln|z^* - z_0|, \quad \forall z^* \in S^*, \tag{3.4}$$

and $v^*(z^*) \equiv v^*(x^*, y^*)$ is a function, harmonically conjugate to $u^*(x^*, y^*)$.

If in (3.2), (3.3), (3.4), instead of z^* we substitute its value from (3.1) and take into account the fact that conformal mapping does not change the analiticity (harmonicity) of a function, then it is clear that the obtained function

$$f(z) = \frac{1}{z - z_0} \exp\left[u(z) + iv(z)\right] \tag{3.5}$$

realizes conformal mapping of the infinite domain D onto the disk G. In formula (3.5), for the function $u(z)$ (regular at infinity) we obtain the boundary problem

$$\Delta u(z) = 0, \quad \forall z \in D, \tag{3.6}$$

$$u(z) = \ln|z - z_0|, \quad \forall z \in S, \tag{3.7}$$

$$u(z) = c + O\left(\frac{1}{|z|}\right) \quad \text{for} \quad |z| \to \infty, \tag{3.8}$$

where c is the real constant and $|c| < \infty$, $v(z) \equiv v(x,y)$ is a function, harmonically conjugate to $u(x,y)$, which is found to within an arbitrary constant sammand [1], [2]:

$$v(x,y) = \int\limits_{(x_0,y_0)}^{(x,y)} \left(\frac{\partial u}{\partial x}\, dy - \frac{\partial u}{\partial y} dx\right) + \alpha, \tag{3.9}$$

where $(x,y) \in D$, and (x_0,y_0) is the fixed point in the domain D.

The line integral in (3.9) does not depend on the way of integration and the constant α implies only rotation of the mapping by the angle α and hence this constant may always be chosen so that the fixed point $z_1 \in S$ will be transformed upon mapping into the point $\zeta_1(1,0) \in \gamma$. Since the normalization conditions $f(\infty) = (0,0)$, $f(z_1) = \zeta_1$ are fulfilled, the function $f(z)$ is defined uniquely.

We seek for an approximate solution of problem (3.6), (3.7), (3.8) in the form (2.7). Consequently, the harmonic function $v_N(x,y)$, conjugate to $u_N(x,y)$, will have the form

$$v_N(z) \equiv v_N((x,y) = \sum_{k=1}^{N} a_k^{(N)} \arg\left[\frac{\widetilde{z}_k - z}{(z - z_0)(\widetilde{z}_k - z_0)}\right] + \alpha_N, \tag{3.10}$$

where α_N is an arbitrary constant.

In (3.5), instead of the functions $u(z)$ and $v(z)$ we take $u_N(z)$ and $v_N(z)$, respectively, The obtained function has the form

$$f_N(z) = \frac{1}{z - z_0} \exp\left[u_N(z) + iv_N(z)\right], \quad z \in \overline{D}. \tag{3.11}$$

It is evident from the construction that the function $f_N(z)$ is holomorphic in $\overline{D}$ (since the Cauchy-Riemann conditions are fulfilled for it), i.e., it is single-valued, continuous and continuously differentiable n times in $\overline{D}$.

It is easily seen that if the boundary problem (3.6), (3.7), (3.8) is solved to within $\varepsilon (0 < \varepsilon < 1)$, i.e., $|u(z) - u_N(z)| < \varepsilon$, $\forall z \in S$, then for the modulus of the function $f_N(z)$ the estimate

$$e^{-\varepsilon} < \left|f_N(z)\right| < e^{\varepsilon}, \quad \forall z \in S, \tag{3.12}$$

is valid. In terms of formulas (2.11) we have

$$\lim_{N\to\infty} v_N(z) = v(z), \quad z \in D,$$
$$\lim_{N\to\infty} f_N(z) = f(z), \quad z \in D,$$

and while solving the problems of practical nature by using the method of conformal mapping, we can assume that $\lim\limits_{N\to\infty} f_N(z) = f(z)$ for $z \in \overline{D}$.

Indeed, in practical problems the contour S has a tolerance field, we can take the domain D_1 with boundary S_1 as plysical domain D. Here the contour $S_1 \in D$ and the deviation $\delta = \max \rho(S, S_1)$ of the contour S_1 from S belongs to the tolerance field. Thus on the basis of the continuity of conformal mapping and correctness of problem (3.6), (3.7), (3.8), in practical problems we may assume that $\lim\limits_{N\to\infty} v_N(z) = v(z),\ z \in \overline{D}$ and, consequently, $\lim\limits_{N\to\infty} f_N(z) = f(z),\ z \in \overline{D}$.

It can be easily shown that if the relation $|v_N(z) - v(z)| < \delta$, $z \in S$ for the given $\varepsilon > 0$ and N is valid, then

$$\left|f_N(z) - f(z)\right| < \sqrt{\varepsilon^2 + e^{\varepsilon}\delta^2}, \quad z \in S.$$

If in (3.10) we take

$$\alpha_N = -\sum_{k=1}^{N} a_k^{(N)} \arg\left[\frac{\widetilde{z}_k - z}{(z - z_0)(\widetilde{z}_k - z_0)}\right] + \arg(z_1 - z_0),$$

then $f_N(\infty) = 0$, $f_N(z_1) = \zeta_1 = (1, 0)$, i.e., the normalitation conditions are fulfilled.

While constructing the function $f_N(z)$, practically, we rely on Osgood's theorem [12], [13]. On the basis of this theorem, if $\arg f_N(z)$ increases strictly monotonically from 0 to 2π at a single circuit around the line S (clockwise, starting from z_1), then the function $f_N(z)$ provides conformal mapping of the domain D onto the disk G to writhin $\varepsilon_1 = e^{\varepsilon} - 1$. If the above-mentioned condition is not fulfilled, then the accuracy of solving the boundary value problem (3.6), (3.7), (3.8) must be increased.

4. Numerical examples

The tables based on experimental data illustrate the dynamics of accuracy of the solition of problem (3.6), (3.7), (3.8) (or relying on the inequality (3.12), the dynamics of accuracy of constucting the function $\zeta = f_N(z)$) depending on a choice of the auxiliary contour $\widetilde{S}$ and N when auxiliary points

$\widetilde{z}_k$ and collocation points z_k $(k = 1, 2, \dots, N)$ lie "uniformly" on the contours $\widetilde{S}$ and S, respectivly. In all examples considered below, the coefficients $a_k^{(N)}$ of an approximate solution $u_N(z)$ of problem (3.6), (3.7), (3.8) are defined from system (2.9) by using the collocation method.

In the tables, $\varepsilon_{\max}$ is the a posteriori estimate of the solution error of problem (3.6), (3.7), (3.8),

$$\varepsilon_{max} = \max_{z \in S} \left| g(z) - u_N(z) \right| \approx \max \left| g(z_j) - u_N(z_j) \right|,$$

where $g(z) = \ln|z - z_0|$ $(z_0 \in B)$, the points z_j $(j = 1, 2, \dots, M)$ lie "uniformly" on the boundary S and $M >> N$. In nymerical experiments we have taken $z_0 = 0$, $M = 10000$ and computations were performed with double accuracy.

Example 1. D is an exterior of the ellips $S : x = a \cos t$, $y = b \sin t$ $(0 \le t \le 2\pi)$, and the auxiliary contour is the ellipse $\widetilde{S} : \widetilde{x} = a_1 \cos t$, $\widetilde{y} = b_1 \sin t (0 \le t \le 2\pi)$. In numerical experiments, the points z_k and $\widetilde{z}_k$ $(k = 1, 2, \cdots, N)$ lie uniformly with respect to the parameter t on contours S and $\widetilde{S}$, respectively. Analogously, for calculation of $\underset{\varepsilon}{max}$ we have taken the points z_j $(j = 1, 2, \cdots, M)$ uniformly lying on S with respect to t. In Tables 1,2,3,4 give the results of numerical experiments for various values of a, b, a_1, b_1 and N.

Table 1

$a = 5\,;\ b = 2\ ; a_1 = a - \delta\ ; b_1 = b - \delta$				
δ	N=100 ε_{max}	N=200 ε_{max}	N=300 ε_{max}	N=400 ε_{max}
0.1	$0.3E-02$	$0.2E-03$	$0.2E-04$	$0.1E-04$
0.2	$0.4E-03$	$0.3E-05$	$0.6E-02$	$0.2E-03$
0.3	$0.5E-04$	$0.7E-01$	$0.3E-04$	$0.1E-05$
0.4	$0.7E-03$	$0.5E-00$	$0.4E-05$	$0.3E-07$
0.5	$0.6E-01$	$0.6E-01$	$0.5E-05$	$0.7E-08$
$a = 5\,; b = 2\ ; a_1 = a - \delta\ ; b_1 = b - 2\delta$				
0.1	$0.4E-03$	$0.3E-05$	$0.4E-07$	$0.6E-09$
0.2	$0.8E-05$	$0.1E-08$	$0.2E-12$	$0.1E-11$
0.3	$0.1E-06$	$0.1E-11$	$0.2E-10$	$0.1E-09$
0.4	$0.3E-07$	$0.1E-09$	$0.5E-11$	$0.2E-13$
0.5	$0.2E-05$	$0.1E-06$	$0.1E-11$	$0.2E-11$
$a = 5\ ; b = 2\ ; a_1 = a - \delta\ ; b_1 = b - 3\delta$				
0.1	$0.8E-04$	$0.2E-06$	$0.5E-09$	$0.2E-11$
0.2	$0.6E-06$	$0.5E-12$	$0.3E-14$	$0.3E-14$
0.3	$0.2E-08$	$0.3E-14$	$0.3E-14$	$0.8E-14$
0.4	$0.2E-08$	$0.6E-14$	$0.1E-13$	$0.4E-13$
0.5	$0.2E-07$	$0.1E-11$	$0.4E-11$	$0.3E-11$

Table 2

$a=5;\ b=1;a_1=a-0.05;b_1=b-\delta$					
δ	N=100 ε_{max}	N=200 ε_{max}	N=300 ε_{max}	N=400 ε_{max}	N=500 ε_{max}
0.3	$0.3E-03$	$0.4E-06$	$0.6E-09$	$0.1E-11$	$0.4E-14$
0.5	$0.7E-05$	$0.2E-08$	$0.6E-12$	$0.5E-12$	$0.6E-10$
0.7	$0.5E-05$	$0.2E-08$	$0.2E-10$	$0.2E-07$	$0.2E-08$
0.9	$0.7E-05$	$0.3E-08$	$0.3E-09$	$0.4E-09$	$0.7E-10$
$a=10;b=1;a_1=a-0.05;b_1=b-\delta$					
0.3	$0.5E-02$	$0.1E-03$	$0.5E-05$	$0.2E-05$	$0.6E-05$
0.5	$0.3E-03$	$0.3E-05$	$0.2E-07$	$0.7E-09$	$0.5E-10$
0.7	$0.7E-04$	$0.4E-06$	$0.2E-08$	$0.1E-10$	$0.1E-14$
0.9	$0.7E-05$	$0.3E-07$	$0.2E-09$	$0.1E-11$	$0.6E-14$
$a=20;b=1;a_1=a-0.01;b_1=b-\delta$					
0.3	$0.2E-01$	$0.2E-02$	$0.4E-03$	$0.7E-04$	$0.1E-04$
0.5	$0.3E-02$	$0.2E-03$	$0.3E-04$	$0.4E-05$	$0.8E-06$
0.7	$0.1E-02$	$0.2E-03$	$0.1E-04$	$0.3E-05$	$0.2E-06$
0.9	$0.8E-03$	$0.2E-03$	$0.9E-05$	$0.4E-05$	$0.9E-06$
$a=40;b=1;a_1=a-0.01;b_1=b-\delta$					
0.3	$0.2E-01$	$0.9E-02$	$0.3E-02$	$0.1E-02$	$0.5E-03$
0.5	$0.6E-02$	$0.1E-02$	$0.2E-03$	$0.9E-04$	$0.3E-03$
0.7	$0.5E-02$	$0.4E-03$	$0.7E-04$	$0.2E-04$	$0.5E-05$
0.9	$0.4E-02$	$0.3E-03$	$0.2E-04$	$0.4E-05$	$0.2E-05$

Example 2. Let us consider the conformal mapping of the exterior of astroid $S: x=2\cos^3 t,\ y=2\sin^3 t(0\le t\le 2\pi)$ onto the disk $|\zeta|<1$. The asteroid $\widetilde{S}: \widetilde{x}=(2-\delta)\cos^3 t,\ \widetilde{y}=(2-\delta)\sin^3 t(0\le t\le 2\pi)$ was taken as an auxiliary contour. In Table 4, the numerical results are given for various δ and N.

Table 3

$a=40;\ b=0.5;a_1=a-0.001;b_1=b-\delta$				
δ	N=700 ε_{max}	N=800 ε_{max}	N=900 ε_{max}	N=1000 ε_{max}
0.15	$0.35E-03$	$0.23E-03$	$0.15E-03$	$0.11E-03$
0.25	$0.98E-04$	$0.64E-04$	$0.46E-04$	$0.31E-04$
0.35	$0.12E-03$	$0.78E-04$	$0.48E-04$	$0.24E-04$
0.45	$0.12E-03$	$0.73E-04$	$0.36E-04$	$0.11E-04$

Example 3. We take conformal mapping of the exterior of the square $D: |x|>1,\ |y|>1$. In this case we take a contour which is obtained from the initial contour by pressing δ times (i.e., $\widetilde{x}=\delta x,\ \widetilde{y}=\delta y$, where $(\widetilde{x},\widetilde{y})\in\widetilde{S}$

and $(x, y) \in S$) in the role of $\widetilde{S}$. The results for various δ and N are given in Table 5.

Table 4

N	δ=0.0001 ε_{max}	δ=0.00001 ε_{max}	δ=0.000001 ε_{max}
100	$0.55E-01$	$0.76E-01$	$0.96E-01$
200	$0.30E-01$	$0.37E-01$	$0.47E-01$
300	$0.23E-01$	$0.24E-01$	$0.30E-01$
400	$0.19E-01$	$0.17E-01$	$0.23E-01$
500	$0.18E-01$	$0.14E-01$	$0.18E-01$
600	$0.17E-01$	$0.11E-01$	$0.15E-01$
700	$0.18E-01$	$0.94E-02$	$0.12E-01$
800	$0.21E-01$	$0.81E-02$	$0.11E-01$
900	$0.40E-01$	$0.71E-02$	$0.95E-02$
1000	$0.74E-01$	$0.63E-02$	$0.85E-02$

Table 5

N	δ=0.997 ε_{max}	δ=0.999 ε_{max}	δ=0.9995 ε_{max}
100	$0.31E-01$	$0.50E-01$	$0.62E-01$
200	$0.11E-01$	$0.24E-01$	$0.31E-01$
300	$0.87E-02$	$0.15E-01$	$0.21E-01$
400	$0.85E-02$	$0.10E-01$	$0.15E-01$
500	$0.84E-02$	$0.72E-02$	$0.12E-01$
600	$0.84E-02$	$0.53E-02$	$0.94E-02$
700	$0.84E-02$	$0.39E-02$	$0.76E-02$
800	$0.84E-02$	$0.38E-02$	$0.63E-02$
900	$0.84E-02$	$0.37E-02$	$0.53E-02$
1000	$0.84E-02$	$0.36E-02$	$0.44E-02$

Example 4. Consider the conformal mapping of the exterior of Booth's lemniscate $S : z = \frac{\zeta}{1-a\zeta^2}$, $0 < a < 1$ and $\zeta = e^{i\theta}(0 \le \theta \le 2\pi)$ onto the disk $|\zeta| < 1$. Since the function $z = \frac{\zeta}{1-a\zeta^2}$ realizes conformal mapping of the unit disk $|\zeta| < 1$ onto the interior of the above-mentioned lemniscate, in the role of the contour $\widetilde{S}$ we can take the contour $\widetilde{S} : \widetilde{z} = \frac{\widetilde{\zeta}}{1-a\widetilde{\zeta}^2}$, where $\widetilde{\zeta} = (1-\delta)e^{i\theta}(0 < \delta < 1, 0 \le \theta \le 2\pi)$. The results for various a, δ and N are given in Tables 6 and 7.

Example 5. Let D be the exterior of Pascal's limacon S: $x = R(cos\theta + a\cos 2\theta)$, $y = R(sin\theta + a\sin 2\theta)$ where $0 \le a \le 1/2$, $0 \le \theta \le 2\pi$ and $R > 0$. Since the function $z = R(\zeta + a\zeta^2)$ maps conformally the unit disk $|\zeta| < 1$

onto the interion of Pascal's limacon, in the role of $\widetilde{S}$ we may take the contour $\widetilde{z} = R(\widetilde{\zeta} + a\widetilde{\zeta}^2)$, where $\widetilde{\zeta} = (1-\delta)e^{i\theta}(0 \leq \theta \leq 2\pi)$, $0 < \delta < 1)$. The numerical experiments were carried out for $a = 0.5$ and $R = 2$. It should be noted that for $a = 1/2$ the contour S has a cusp $-(-R/2, 0)$. The numerical results for various δ and N are given in Table 8.

a=0.5 Table 6

N	δ=0.1 ε_{max}	δ=0.3 ε_{max}	δ=0.5 ε_{max}
100	$0.1E-03$	$0.6E-05$	$0.1E-05$
200	$0.2E-06$	$0.2E-12$	$0.9E-08$
300	$0.3E-10$	$0.3E-13$	$0.4E-08$
400	$0.3E-13$	$0.2E-13$	$0.4E-08$
500	$0.5E-14$	$0.2E-13$	$0.3E-08$
600	$0.1E-13$	$0.3E-13$	$0.1E-07$

a=0.9 Table 7

N	δ=0.01 ε_{max}	δ=0.03 ε_{max}	δ=0.05 ε_{max}
100	$0.3E+00$	$0.9E-01$	$0.4E-01$
200	$0.9E-01$	$0.2E-01$	$0.6E+00$
300	$0.4E-01$	$0.3E-02$	$0.2E+00$
400	$0.2E-01$	$0.5E-03$	$0.5E-03$
500	$0.9E-02$	$0.8E-04$	$0.6E-02$
600	$0.5E-02$	$0.1E-04$	$0.6E-03$
700	$0.2E-02$	$0.2E-05$	$0.3E-05$
800	$0.1E-02$	$0.4E-06$	$0.1E-07$
900	$0.6E-03$	$0.7E-07$	$0.5E-10$
1000	$0.3E-03$	$0.1E-07$	$0.3E-11$

a=0.5 Table 8

N	δ=0.05 ε_{max}	δ=0.1 ε_{max}	δ=0.2 ε_{max}	δ=0.4 ε_{max}
100	$0.2E-03$	$0.9E-06$	$0.2E-07$	$0.2E-07$
200	$0.6E-06$	$0.1E-10$	$0.7E-11$	$0.1E-07$
300	$0.2E-08$	$0.5E-13$	$0.7E-13$	$0.1E-07$
400	$0.1E-10$	$0.3E-14$	$0.3E-12$	$0.2E-07$
500	$0.5E-13$	$0.3E-14$	$0.2E-12$	$0.1E-07$
600	$0.3E-14$	$0.3E-14$	$0.5E-11$	$0.7E-07$

Example 6. Let D be the exterior of the epicycloid $S : z = \omega(\zeta) = R(\zeta + \zeta^4/4)$, where $\zeta = e^{i\theta}(0 \leq \theta \leq 2\pi,\ R > 0)$. Since the function $z = \omega(\zeta)$ maps conformally the unit disk $|\zeta| < 1$ onto the interior of the

epicycloid, in the role of $\widetilde{S}$ we may take the contour $\widetilde{z} = R(\widetilde{\zeta} + \widetilde{\zeta}^4/4)$, where $\widetilde{\zeta} = (1-\delta)e^{i\theta}$, $0 < \delta < 1$. The results of numerical experiments for various δ and N for $R = 2$ are given in Table 9.

Table 9

N	δ=0.05 ε_{max}	δ=0.1 ε_{max}	δ=0.2 ε_{max}
100	$0.3E-03$	$0.2E-05$	$0.1E-05$
200	$0.8E-06$	$0.7E-08$	$0.7E-08$
300	$0.3E-08$	$0.5E-10$	$0.7E-10$
400	$0.1E-10$	$0.2E-11$	$0.4E-09$
500	$0.8E-13$	$0.6E-13$	$0.3E-07$
600	$0.5E-14$	$0.3E-14$	$0.1E-07$

5. Comments

In Aleksidze's monograph [8] it is shown that while solving the Dirichlet plane boundary value problem for the Laplace equation by using MFS the following facts take place: (1) for fixed N there exists an optimal auxiliary contour $\widetilde{S}$ (or optimal location of $\widetilde{z}_n$ points) in the sense of approximation accuracy of the boundary function; (2) if an auxiliary contour $\widetilde{S}$ moves away from the basic boundary S, then the conditionality of the system matrix (2.9) deteriorates and, on the contrary, improves if $\widetilde{S}$ approaches S, but accuracy of the solution deteriorates (in this case to achieving the necessary accuracy it is sufficient to increase N).

Numerical experiments have shown that: (3) for the fixed N the theoretically existing optimal auxiliary contour $\widetilde{S}$ has a neighborhood in which all contours satisfying the conditions stated in Remark 1 give satisfactory accuracy; (4) the width of such strip and its nearness to the contour S depends both on the differential and simple geometrical properties of the contour S and on the boundary function.

It should be noted that in view of a plysical sense of fundamental solutions [7], [8], [9], while solving the boundary value problems for domains with reentrant angles there arise difficulties of purely technical character (this is expressed by the fact that in this case the systems of higher dimensions are to be solved; see Examples 2 and 3). In this case the contour $\widetilde{S}$ approaches S and therefore to obtain good results the number N should be increased after which the system matrix (2.9) automatically becomes well agreed upon.

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