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CLASSES OF PSEUDO-CONCAVE TYPE FUNCTIONS AND THEIR APPLICATIONS

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Abstract. In the present paper we introduce classes of functions of pseudo-concave–pseudo-convex type functions $(A) : A^{\gamma(t)}$, $A_{\lambda(t)}$, $A^{\gamma(t)}_{\lambda(t)}$ and $(W) : W^{\gamma(t)}$, $W_{\lambda(t)}$, $W^{\gamma(t)}_{\lambda(t)}$, respectively, with functional parameters. A complete description of such classes and some of their applications are given. Their connection with the well-known Barry-Stechkin type classes Φ_k corresponding to $\gamma(t) = 1$, $\lambda(t) = t^k$ $k = 1, 2, \dots,$ and with other majorant classes is established.

1. Main Definitions and Notation

Definition 1. A nonnegative function $f(x)$ is said to be almost increasing (almost decreasing) on a set $E \subseteq$, if the exists a constant $C > 0$ $(d > 0)$ such that for all x, $y \in E$ from the inequality $x < y$ follows the inequality $f(x) \le Cf(y)$ $(f(x) \ge df(y))$. The number

$$C(f) = \sup_{x<y;\, x,y\in E,\, f(y)\neq 0} \frac{f(x)}{f(y)}, \quad 1 \le C(f) < \infty, \tag{1}$$

is called a coefficient of almost increase of an almost increasing function $f(x)$. Similarly,

$$d(f) = \inf_{x<y;\, x,y\in E,\, f(y)\neq 0} \frac{f(x)}{f(y)}, \quad 0 < d(f) \le 1, \tag{2}$$

is the coefficient of almost decrease of an almost decreasing function (see [1]–[3]).

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Two nonnegative functions $f(x)$ and $\varphi(x)$, $x \in [0,\ell]$ will be called, as usual (see [1]), equivalent on $[0,\ell] : f \sim \varphi$, if there exist $C_1 > 0$ and $C_2 > 0$ such that $C_{1\varphi}(x) \le f(x) \le C_{2\varphi}(x)$.

Lemma 1. *For the function $f(x)$, $0 < x \le \ell$ to be almost increasing (almost decreasing), it is necessary and sufficient that it be equivalent on $[0,\ell]$ to a non-decreasing (non-increasing) function (see [3]).*

Everywhere below, all the functions under consideration will be defined on the interval $[0,\ell]$, $\ell > 0$.

Introduce the following notation:

$W_+ = \{\varphi : \varphi(t) \in C(0,\ell], \varphi(t) > 0, t > 0,$

$W^* = \{\varphi : \varphi(t) \in W_+$ and $\varphi(t)$ almost increases on $(0,\ell\}$,

$W_* = \{\varphi : \varphi(t) \in W_+$ and $\varphi(t)$ almost decreases on $(0,\ell\}$,

$W = W^* \cup W_*$ is a class of almost monotone functions,

$W_*^* = W^* \cup W_*$ is a class of almost simultaneously increasing and decreasing functions,

$W_0^* = \{\varphi : \varphi(t) \in W^*$ and $\varphi(0) = 0\} = W^* \backslash W_*^*$,

$W_\infty = \{\varphi : \varphi(t) \in W_*$ and $\lim\limits_{t\to 0}\varphi(t) = \infty = \infty\} = W_* \backslash W_*^*$.

Let $\alpha(t)$ and $\beta(t)$ be the fixed functions from the class W.

Definition 2. We shall say that the nonnegative function $\varphi(t)$ belongs to the class $A^{\alpha(t)}$ if $\varphi(t)/\alpha(t)$ almost increases and to the class $A_{\beta(t)}$ if $\varphi(t)/\beta(t)$ decreases. By $A_{\beta(t)}^{\alpha(t)}$ we denote the intersection

$$A_{\beta(t)}^{\alpha(t)} = A^{\alpha(t)} \cap A_{\beta(t)}.$$

Classes of the type (A): $A^{\alpha(t)}$, $A_{\beta(t)}$ and $A_{\beta(t)}^{\alpha(t)}$. Basic Properties.

Property 1. *Let $\alpha_i(t) \in W$, $i = 1,2$. If $\frac{\alpha_1(t)}{\alpha_2(t)} \in W^*$, then $A^{\alpha_1(t)} \subset A^{\alpha_2(t)}$. Analogously, if $\frac{\beta_1(t)}{\beta_2(t)} \in W^*$, $\beta_i(t) \in W$, then $A_{\beta_2(t)} \subset A_{\beta_1(t)}$.*

Corollary 1. *$A^{\alpha_1(t)} \equiv A^{\alpha_2(t)}$ and $A_{\beta_1(t)} \equiv A_{\beta_2(t)}$, if $\alpha_1(t) \sim \alpha_2(t)$ and $\beta_1(t) \sim \beta_2(t)$.*

By Lemma 1 and Corollary 1, the functions $\alpha(t)$ and $\beta(t)$ can be assumed to be monotone.

Property 2. *If $f(x) \in A_{\beta(x)}^{\alpha(x)}$, then there exist positive constants d and C such that*

$$d\beta(x) \le f(x) \le C\alpha(x).$$

In the case of the classes $A^{\alpha(t)}$ and $A_{\beta(t)}$ the corresponding one-sided inequalities are valid.

Following (1) and (2), we introduce the notation:

$$C_\alpha(f) = \sup_{0<x<y\le\ell} \frac{\alpha(y)f(x)}{\alpha(x)f(y)}, \quad \text{for} \quad f(x) \in A^{\alpha(x)} \tag{3}$$

and

$$d_\beta(f) = \inf_{0<x<y\le\ell} \frac{\beta(y)f(x)}{\beta(x)f(y)}, \quad \text{for} \quad f(x) \in A_{\beta(x)}. \tag{4}$$

Property 3. *If $f(x) \in A^{\alpha_1(x)}$ and $g(x) \in A^{\alpha_2(x)}$, $\alpha_i(x) \in W$, $i = 1, 2$, then*

$$f(x)g(x) \in A^{\alpha_1(x)\alpha_2(x)} \text{ and } C_{\alpha_1\alpha_2}(fg) \le C_{\alpha_1}(f)C_{\alpha_2}(g). \tag{5}$$

The similar statement will be true for the classes $A_{\beta(x)}$ if we replace (5) by $d_{\beta_1\beta_2}(fg) \ge d_{\beta_1}(f)d_{\beta_2}(g)$.

Property 4. *For the class $A^{\alpha(x)}_{\beta(x)}$ to be empty, it is necessary and sufficient that $\frac{\beta(x)}{\alpha(x)}$ be almost increasing.*

Property 5. *The class $A^{\alpha(t)}_{\beta(t)}$ possesses the property*

$$f(x) \in A^{\alpha(x)}_{\beta(x)} \Leftrightarrow f(x)\delta(x) \in A^{\alpha(x)\delta(x)}_{\beta(x)\delta(x)}$$

for any functions $\alpha(x)$, $\beta(x)$, $\delta(x) \in W$.

Property 6. *If $f(x) \in A^{\alpha(x)}$, then $1/f(x) \in A_{1/\alpha(x)}$, where $d_\alpha(1/f) = [C_\alpha(f)]^{-1}$. Similarly, if $f(x) \in A_{\beta(x)}$, then $1/f(x) \in A^{1/\beta(x)}$ and $C_\beta(1/f) = [d_\beta(f)]^{-1}$.*

Property 7. *If $f(x) \in A^{\alpha(x)}$ or $f(x) \in A_{\beta(x)}$, then $f^{-1}(x) \in A_{\alpha^{-1}(t)}$ or $f^{-1}(x) \in A^{\beta^{-1}(t)}$, respectively.*

Property 8. *Let $f(x) \in A^{\alpha(x)}$, $f(x) \ne 0$ for $0 < x < \ell$ and the function $1/f(x)$ be integrable in the neighborhood of the origin. Then $\alpha(x)/x \in W_\infty$ and*

$$h_f(x) \stackrel{\text{def}}{=} \int_0^x \frac{dt}{f(t)} \in A^0_{x/\alpha(x)},$$

where $d_{x/\alpha(x)}(h_f) \ge 1/C_\alpha(f)$.

Property 9. *Let $f(x) \in A^{\alpha(x)}_{\beta(x)}$, where $\alpha(x)$ and $\beta(x)$ are non-decreasing and tending to $k+\infty$ as $x \to +\infty$. Then*

$$f\big[h_f^{-1}\big] \in A^{\alpha[\frac{x}{\alpha(x)}]}_{\beta[\frac{x}{\beta(x)}]}.$$

Proofs of Properties 1–9 are obtained by a direct reasoning on the basis of the definition of classes $A^{\alpha(x)}_{\beta(x)}$.

Isomorphism of Classes $A^{\alpha(x)}$, $A_{\beta(x)}$, $A^{\alpha(x)}_{\beta(x)}$. In the lemma and theorem below, the functions $\alpha_k(x)$ and $\beta_k(x)$, $k=1,2$, are assumed to be fixed and monotone of the class $W_+([0,\ell])$.

Lemma 2. *The classes $A^{\alpha_1(x)}$ and $A^{\alpha_2(x)}$ are isomorphic. The classes $A_{\beta_1(x)}$ and $A_{\beta_2(x)}$ are isomorphic, as well. The isomorphism between these classes is prescribed by the relations*

$$f_1(x)=\frac{\alpha_1(x)}{\alpha_2(x)}\,f_2(x)\ \ \text{and}\ \ f_1(x)=\frac{\beta_1(x)}{\beta_2(x)}\,f_2(x),$$

respectively.

Theorem 3. *The isomorphism between the classes $A^{\alpha_1(x)}_{\beta_1(x)}$ and $A^{\alpha_2(x)}_{\beta_2(x)}$ is prescribed by the relation*

$$f_1(x)=\frac{\alpha_1(x)}{[\alpha_2(x)]^{P(x)}}\,\big[f_2(x)\big]^{P(x)}, \tag{6}$$

where

$$P(x)=\ln\frac{\beta_1(x)}{\alpha_1(x)}\Big/\ln\frac{\beta_2(x)}{\alpha_2(x)},\quad \frac{\beta_i(x)}{\alpha_i(x)}\in W_0^*,\ \ i=\overline{1,2}.$$

Proof. From (6) we have

$$\Big[\frac{f_1(x)}{\alpha_1(x)}\Big]^{\ln\frac{\beta_2(x)}{\alpha_2(x)}}=\Big[\frac{f_2(x)}{\alpha_2(x)}\Big]^{\ln\frac{\beta_1(x)}{\alpha_1(x)}}.$$

Thus the almost increase of the function $\frac{f_2(x)}{\alpha_2(x)}$ implies the almost increases of the function $\frac{f_1(x)}{\alpha_1(x)}$. Similarly,

$$\frac{f_1(x)}{\beta_1(x)}=\frac{\alpha_1(x)}{\beta_1(x)}\Big[\frac{f_2(x)}{\beta_2(x)}\Big]^{P(x)}\cdot\Big[\frac{\beta_2(x)}{\alpha_2(x)}\Big]^{P(x)}.$$

In order for $\frac{f_1(x)}{\beta_1(x)}$ almost to decrease, it is sufficient that $\frac{\alpha_1(x)}{\beta_1(x)}\cdot[\frac{\beta_2(x)}{\alpha_2(x)}]^{P(x)}=1$. This can be achieved by taking logarithm in both parts of that equality. □

The most natural is the situation $\alpha(t)=t^\alpha$, $\beta(t)=t^\beta$, $\alpha,\ \beta\in R$. We accept the notation $A^{t^\alpha}=A^\alpha$, $A_{t^\beta}=A_\beta$, $A^\alpha_\beta=A^\alpha\cap A_\beta$.

Note that the coefficients $C_\alpha(t)$ and $d_\beta(t)$ (see (3), (4)) increase monotonically with respect to parameters α and β:

$$C_{\alpha_1}(f)\le C_{\alpha_2}(f)\ \ \text{and}\ \ d_{\beta_1}(f)\le d_{\beta_2}(f)\ \ \text{for}\ \ \alpha_1\le\alpha_2\ \ \text{and}\ \ \beta_1\le\beta_2.$$

Obviously, every function $f(x)\in A^\alpha$ is integrable in case $\alpha>-1$. The following inverse statement is also valid.

Lemma 4. *If $f(x) \not\equiv 0$, $f(x) \in A_\beta$ and for any $d > 0$*

$$\int_0^d f(t)\,dt < \infty,$$

then $\beta > -1$.

From Property 2 follows

Lemma 5. *If $f(x) \in A_\beta^\alpha$, then*

$$d_\beta(f)\lambda^\beta f(x) \le f(\lambda x) \le C_\alpha(f)\lambda^\alpha f(x) \ \text{ for } \ 0 \le \lambda \le 1 \tag{7}$$

and

$$\frac{1}{C_\alpha(f)}\lambda^\alpha f(x) \le f(\lambda x) \le \frac{1}{d_\beta(f)}\lambda^\beta f(x) \ \text{ for } \ \lambda \ge 1. \tag{8}$$

In the case of classes A^α, A_β in (7) *and* (8) *the one-sided inequalities corresponding to the classes A^α and A_β are valid, respectively.*

Lemma 6. *Let $f(x) \in A^\alpha$, $\alpha > 0$ and $f(x)$ strictly increase. Then $f^{-1}(x) \in A_{1/\alpha}$, where $f^{-1}(x)$ is the inverse to $f(x)$ function. Analogously, if $f(x) \in A_\beta$, $\beta > 0$ and $f(x)$ is strictly monotone, then $f^{-1}(x) \in A^{1/\beta}$. Moreover,*

$$C_{1/\beta}(f^{-1}) = \big[d_\beta(f)\big]^{-1/\beta} \ \text{ and } \ d_{1/\alpha}(f^{-1}) = \big[C_\alpha(f)\big]^{-1/\alpha}. \tag{9}$$

Proof. The statements of the lemma and equality (9) (see (3),(4)) can be verified directly. □

Lemma 7. *Let $f(x) \in A_\nu^\mu$, $g(x) \in A_\beta^\alpha$. $\beta \ge \alpha \ge 0$, $\nu \ge \mu \ge 0$. Then*

$$f \circ g \overset{\text{def}}{=} f\big[g(x)\big] \in A_{\beta\nu}^{\alpha\mu},$$

and

$$C_{\alpha\mu}(f \circ g) \le \frac{C_\mu(f)}{d_\nu(f)}\big[C_\alpha(g)\big]^\mu \big[C_0(g)\big]^{\nu-\mu}, \tag{10}$$

$$d_{\beta\nu}(f \circ g) \ge \frac{d_\nu(f)}{C_\mu(f)}\big[d_\beta(g)\big]^\nu \big[C_0(g)\big]^{\mu-\nu}. \tag{11}$$

The above lemma can be proved directly by simple transformations.

Lemma 8. *Let $f(x) \in A_\beta^\alpha$ for $0 \le \alpha \le \beta < 1$, $f(x) \ne 0$ and let $0 < x < \ell$ be the function introduced in Property* 8, *and $ah_f^{-1}(x)$ be the inverse to it function. Then*

$$f \circ h_f^{-1} \overset{\text{def}}{=} f\big[h_f^{-1}(x)\big] \in A_{\frac{\beta}{1-\beta}}^{\frac{\alpha}{1-\alpha}},$$

note that

$$C_{\frac{\delta}{1-\alpha}}\big(f \circ h_f^{-1}\big) \le \big[C_\alpha(f)\big]^{\frac{1}{1-\alpha}} / d_\beta(f)$$

and

$$d_{\frac{\beta}{1-\beta}}\left(f\circ h_f^{-1}\right) \geq \left[d_\beta(f)\right]^{\frac{1}{1-\beta}}/C_\alpha(f).$$

Proof. By Lemmas 6 and 7 and also Property 8 we conclude that $1/f(x)$ is integrable in the neighborhood of the origin and $h_f(x) \in A_{1-\alpha}^{1-\beta}$, where

$$C_{1-\beta}(h_f) \leq 1/d_\beta(f), \quad d_{1-\alpha}(h_f) \geq 1/C_\alpha(f).$$

But then by Lemma 6 we have $h_f^{-1}(x) \in A_{\frac{1}{1-\beta}}^{\frac{1}{1-\alpha}}$, and taking into account (9) and inequalities (10) and (11), we obtain

$$C_{\frac{\alpha}{1-\alpha}}\left(f\circ h_f^{-1}\right) \leq \frac{C_\alpha(f)}{d_\beta(f)}\left[C_{\frac{1}{1-\alpha}}(h_f^{-1})\right]^\alpha \leq \frac{[C_\alpha(f)]^{\frac{1}{1-\alpha}}}{d_\beta(f)},$$

$$d_{\frac{\beta}{1-\beta}}\left(f\circ h_f^{-1}\right) \geq \frac{d_\beta(f)}{C_\alpha(f)}\left[d_{\frac{1}{1-\beta}}(h_f^{-1})\right]^\beta \geq \frac{[d_\beta(f)]^{\frac{1}{1-\beta}}}{C_\alpha(f)}. \quad \square$$

On the Class A^∞. Denote

$$A^\infty = \bigcap_{\alpha>0} A^\alpha.$$

The class A^∞ is non-empty: an example of a function from that class is the function

$$f(x) = e^{x-1/x}, \quad x > 0. \tag{12}$$

Lemma 9. *The coefficient C_α of the function* (12) *for any α is calculated by the formula*

$$C_\alpha(f) = e^{-2\sqrt{\alpha^2-4}}\left[\frac{\alpha+\sqrt{\alpha^2-4}}{2}\right]^{2\alpha}, \tag{13}$$

if $\alpha \geq 2$ and $C_\alpha(f) = 1$, if $\alpha \leq 2$.

The proof is carried out by direct calculations: Let $f_\alpha(x) = x^{-\alpha}f(x)$, then $f'_\alpha(x) = e^{x-\frac{1}{x}}\frac{x^2-\alpha x+1}{x^{\alpha+2}}$, from which we easily obtain (13).

Equality (13) shows that $C_\alpha(f)$ increases rapidly as $\alpha \to \infty$ (for any function $f \in A^\infty$, as the lemma shows).

Lemma 10. *For any function $f(x) \in A^\infty$, $f(x) \not\equiv 0$, $\lim\limits_{\alpha\to\infty} C_\alpha(f) = \infty$; moreover, $C_\alpha(f)$ increases as $\alpha \to \infty$ more rapid than the exponential function with arbitrarily large base:*

$$C_\alpha(f) \geq CN^\alpha,$$

no matter how $N > 1$ is; here $C = C(N) > 0$ does not depend on α.

Proof. For any α, by (2) we have

$$C_\alpha(f) = \sup_{0<x\le y} \left(\frac{y}{x}\right)^\alpha \frac{f(x)}{f(y)} \ge \sup_{0<x\le Nx} N^\alpha \frac{f(x)}{f(Nx)} = CN^\alpha,$$

where $C = \sup\limits_{x>0} f(x)/f(Nx) > 0$, if $f(x) \not\equiv 0$. □

Corollary 2. *There is no function $f(x) \ge 0$, except $f(x) \not\equiv 0$, such that $f(x)/x^\alpha$ would increase for any α.*

Indeed, if there is any, then for it $C_\alpha(f) \not\equiv 1$ for all α, which contradicts Lemma 10.

Classes of the Type (W): $W^{\gamma(t)}$, $W_{\lambda(t)}$ and $W^{\gamma(t)}_{\lambda(t)}$.

Definition 3. We shall say that the function $\psi(t) \in W_0^*$, $0 \le t \le \ell$, almost decreases with some "supply" of almost increase, if there exists an almost increasing function $\mu_\varphi(t) \in W_0^*$ such that the function $\psi(t)/\mu_\psi(t)$ almost increases on $(0, \ell]$.

Every such function $\mu_\psi(t)$ is assumed to be called a "supply" of the function $\psi(x)$ almost increase.

Obviously, if $\mu_\psi(t)$ is a "supply" of the function $\psi(t)$ almost increase, then the function $\mu_1(t)$ is such that $\mu_\psi(t)/\mu_1(t)$ almost increases and likewise is a "supply" of the function $\psi(t)$ almost increase.

Definition 4. We shall say that the function $\psi(t)$, $0 < t \le \ell$, almost decreases with some "supply" of almost decrease, if there exists a monotonically increasing function $\nu(t) \in W_0^*$ such that the function $\psi(t)\nu(t)$ almost decreases on $(0, \ell]$.

Every such function $\nu(t)$ we call a "supply" of the function $\psi(t)$ almost decrease and denote it by $\nu_\psi(t)$.

Let $\gamma(t)$ and $\lambda(t)$ be fixed monotone functions of the class W_+.

Definition 5. We shall say that the function $\varphi(t) \in W_+$ belongs to the class $W^{\gamma(t)}$ on $[0, \ell]$ if the function $\varphi(t)/\gamma(t)$ almost increases with some "supply" of almost increase (depending on $\varphi(t)$) and that $\varphi(t)$ belongs to the class $W_{\lambda(t)}$ if the function $\varphi(t)/\lambda(t)$ almost decreases with some "supply" of almost decrease (depending on $\varphi(t)$).

In particular, for $\gamma(t) = t^\gamma$, $\gamma \in \mathbb{R}$ we denote $W^{t^\gamma} \overset{\text{def}}{=} W^\gamma$. Analogously, $W_{t^\lambda} \overset{\text{def}}{=} W_\lambda$. For $\gamma \ge 0$, the class W^γ coincides with the Barry-Stechkin class Φ^γ, and for $\lambda = k$, $k = 1, 2, 3, \dots$, we have the well-known Barry–Stechkin class Φ_k (see [1]–[3], [10]–[13]).

We define the class $W^{\gamma(t)}_{\lambda(t)}$ as the intersection of the classes $W^{\gamma(t)}$ and $W_{\lambda(t)}$:

$$W^{\gamma(t)}_{\lambda(t)} = W^{\gamma(t)} \cap W_{\lambda(t)}.$$

Classes of the type (W) have properties of classes of type (A) with corresponding improvements due to the presence of "supply" functions for the classes (W). For example, if $\gamma(t) \sim \lambda(t)$, then the class $A_{\lambda(t)}^{\gamma(t)}$ consists only of the functions $\varphi(t) \sim \gamma(t) \sim \lambda(t)$, while the class $W_{\lambda(t)}^{\gamma(t)}$ is empty for $\gamma(t) \sim \lambda(t)$. Moreover, the classes (W) have those properties which do not possess the classes (A). For example, the condition $\varphi(t) \in A^{\gamma(t)}$ does not imply the fulfilment of any of the conditions (B_γ)-$(P_g m)$ below, while the condition $\varphi(t) \in W^{\gamma(t)}$ is equivalent to any of the conditions (B_γ)-(P_γ) (see theorems on the equivalence), if for $\mu_\varphi(t)$ and $\nu_\varphi(t)$ the condition (Z_1) is valid.

The inequality in Property 2 for $A_{\beta(t)}^{\alpha(t)}$ in the case of the class $W_{\lambda(t)}^{\gamma(t)}$ is written as follows:

$$C_1 \frac{\lambda(t)}{\nu_f(t)} \le f(t) \le C_2 \gamma(t) \cdot \mu_f(t), \quad \nu_f(t), \quad \mu_f(t) \in W_0^*.$$

In the case of power functions $\gamma(t) = t^\gamma$, $\lambda(t) = t^\lambda$, $\lambda \in R$, $\gamma < \lambda$ we have $C_1 t^{\lambda-\nu} \le f(t) \le C_2 t^{\gamma+\mu}$, ν, $\mu > 0$ etc.

Theorems of the Equivalence. The conditions

$$(Z_1) : \int_0^\delta \frac{\sigma(t)}{t}\, dt \le B_1 \sigma(\delta) \quad \text{and} \quad (Z_2) : \delta \int_\delta^\ell \frac{\sigma(t)}{t^2}\, dt \le B_2 \sigma(\delta), \quad 0 \le \delta,$$

are called, respectively, the first and the second Zygmund condition for $\sigma(\delta)$.

Theorem 11. *For the functions of the class W_+ the following statements are equivalent:*

(C_γ) $\varphi(t) \in W^{\gamma(t)}$ *there exists* $\mu_\varphi(t)$ *such that* (Z_1)*;*

(B_γ) $\sum\limits_{\nu=n+1}^{\infty} \frac{\gamma(\frac{1}{n})}{\gamma(\frac{1}{\nu})} \cdot \frac{\varphi(\frac{1}{\nu})}{\nu} \le \overline{B}_\gamma \varphi(\frac{1}{n})$*;*

(Z_γ) $\int_0^\delta \frac{\gamma(\delta)}{\gamma(t)} \cdot \frac{\varphi(t)}{t}\, dt \le B_\gamma \varphi(\delta)$*;*

(L_γ) $C > 1$,

$$\lim_{\delta \to 0} \frac{\varphi(C\delta)}{\varphi(\delta)} > \xi(C), \quad \xi(C) = \sup_{0\delta<l} \frac{\gamma(C\delta)}{\gamma(\delta)};$$

(S_γ) *for the function* $\varphi(\delta)$ *there exists the power function* $\mu_\varphi(\delta) = \delta^\mu$, $0 < \mu < q_\gamma$ *such that* $\frac{\varphi(\delta)}{\gamma(\delta)\mu_\varphi(\delta)}$ *almost increases,* $q_\gamma = \lim\limits_{t \to 0} \frac{\ln \gamma(t)}{\ln t}$*;*

(P_γ) *for any* $0 < \theta < 1$, *for functions* $\varphi(\delta)$ *there exists* $p > 1$ *such that the inequality*

$$\xi(p)\varphi\Big(\frac{1}{pn}\Big) < \theta\varphi\Big(\frac{1}{n}\Big), \quad \xi(p) = \sup_{0<\alpha<l} \frac{\gamma(p\delta)}{\gamma(\delta)}$$

is valid.

Theorem 12. *For the functions of the class W_+ the following statements are equivalent:*

(C_λ) $\varphi(t) \in W_{\lambda)}$ *and for* $\nu_\varphi(t)$ *is fulfiled* (Z_2);

(B_λ) $\sum\limits_{\nu 1}^{n} \frac{\lambda(\frac{1}{n})}{\lambda(\frac{1}{\nu})} \cdot \frac{\varphi(\frac{1}{\nu})}{\nu} \leq \overline{B}_\lambda \varphi(\frac{1}{n})$;

(Z_λ) $\int\limits_0^l \frac{\lambda(\delta)}{\lambda(t)} \cdot \frac{\varphi(t)}{t}\, dt \leq B_\lambda \varphi(\delta)$;

(L_λ) *for the function* $\varphi(\delta)$ *there exists* $C > 1$ *such that*

$$\lim_{\delta \to 0} = \frac{\varphi(C\delta)}{\varphi(\delta)} < \eta(C), \quad \eta(C) = \inf_{0<\delta<l} \frac{\lambda(C\delta)}{\lambda(\delta)};$$

(S_λ) *for the function* $\varphi(\delta)$ *there exists the power function* $\nu_\varphi(\delta) = \delta^\nu$, $0 < \nu < q_\lambda$ *such that* $\frac{\varphi(\delta)\nu_\varphi(\delta)}{\lambda(\delta)}$ *almost decreases,* $q_\lambda = \lim\limits_{t \to 0} \frac{\ln \lambda(t)}{\ln t}$;

(P_λ) *for any* $0 < \theta < 1$, *for the function* $\varphi(\delta)$ *there exists* $p > 1$ *such that the inequality*

$$\varphi\Big(\frac{1}{n}\Big) < \theta \eta(p) \varphi\Big(\frac{1}{np}\Big), \quad \eta(p) = \inf_{0<\delta<l} \frac{\lambda(p\delta)}{\lambda(\delta)}$$

is valid.

The proofs of these theorems are obtained by means of Lemmas 2 and 3 proven in [1] for cases $\gamma(t) = 1$ and $\lambda(t) = t^k$ $k = 1, 2, \dots$, and by theorems on the isomorphism of classes $W^{\gamma(t)}$ and $W_{\lambda(t)}$.

Classes of the Type $A(\infty)$ and $W(\infty)$. Classes $A^{\gamma(t)}$, $A_{\lambda(t)}$, $A^{\gamma(t)}_{\lambda(t)}$ as well as $W^{\gamma(t)}$, $W_{\lambda(t)}$, $W^{\gamma(t)}_{\lambda(t)}$ reflect a local behavior of functions $\varphi(t)$ from the classes W_+ or W in the neighborhood of zero.

To consider the functions $\varphi(t)$ from the classes W_+ and W in the neighborhood of the point at infinity $+\infty$, we shall impose on the functions $\varphi(t)$ certain additional restrictions, analogous to those introduced for the case in the neighborhood of zero.

Let $\gamma(t)$ and $\lambda(t)$ be the given monotone functions of the class $W(0, \infty)$. We define classes $A^{\gamma(t)}(\infty)$, $A_{\lambda(t)}(\infty)$ and $A^{\gamma(t)}_{\lambda(t)}(\infty)$ as follows.

We shall say that the function $\varphi(t) \in W_+(0, \infty)$ belongs to the class $A^{\gamma(t)}$ or to the class $A_{\lambda(t)}(\infty)$ if $\varphi(t)/\gamma(t)$ almost increases as $t \to \infty$, and there exists $\lim\limits_{t \to \infty} \frac{\varphi(t)}{\gamma(t)} = a$, where $0 < a \leq \infty$ for $\varphi(t) \in A^{\gamma(t)}(\infty)$, and if $\varphi(t)/\lambda(t)$ almost decreases as $t \to \infty$, then there exists $\lim\limits_{t \to \infty} \frac{\varphi(t)}{\lambda(t)} = b$, where $0 \leq b < \infty$ for $\varphi(t) \in A_{\lambda(t)}(\infty)$. The class $A^{\gamma(t)}_{\lambda(t)}(\infty)$ will be defined as an intersection of classes $A^{\gamma(t)}(\infty)$, $A_{\lambda(t)}(\infty) : A^{\gamma(t)}_{\lambda(t)}(\infty) = A^{\gamma(t)}(\infty) \cap A_{\lambda(t)}(\infty)$.

Classes $A^\gamma(\infty)$, $A_\lambda(\infty)$, $A^\gamma_\lambda(\infty)$, γ, $\lambda \in \mathbb{R}$ correspond to the power functions $\gamma(t) = t^\gamma$, $\lambda(t) = t^\lambda$, γ, $\lambda \in \mathbb{R}$.

For definition of the classes $W^{\gamma(t)}(\infty)$, $W_{\lambda(t)}(\infty)$, $W_{\lambda}^{\gamma(t)}(\infty)$ it is more convenient to impose additional conditions in terms of Zygmund conditions.

Let $\gamma(t)$ and $\lambda(t)$ be the given monotone functions of the class $W(0,\infty)$.

Definition 6. We shall say that the function $\varphi(t) \in W_+(0,\infty)$ belongs to the class $W^{\gamma(t)}(\infty)$ or $W_{\lambda(t)}(\infty)$ if for some $\ell > 0$ the integral Zygmund conditions

$$(Z^{\gamma})(\infty): \ \ \sup_{\delta\geq\ell} \frac{\gamma(\delta)}{\varphi(\delta)} \int\limits_{\ell}^{\delta} \frac{\varphi(t)}{t\gamma(t)}\, dt = B_{\gamma}(\infty) < \infty$$

or

$$(Z^{\gamma})(\infty): \ \ \sup_{\delta\geq\ell} \frac{\lambda(\delta)}{\varphi(\delta)} \int\limits_{\delta}^{\infty} \frac{\varphi(t)}{t\lambda(t)}\, dt = C_{\lambda}(\infty) < \infty$$

are respectively fulfilled.

The class $W_{\lambda(t)}^{\gamma(t)}(\infty)$ will be introduced as an intersection of the classes $W^{\gamma(t)}(\infty)$ and $W_{\lambda(t)}(\infty)$:

$$W_{\lambda(t)}^{\gamma(t)}(\infty) = W^{\gamma(t)}(\infty) \cap W_{\lambda(t)}(\infty).$$

The classes $W^{\gamma}(\infty)$, $W_{\lambda}(\infty)$, $W_{\lambda}^{\gamma}(\infty)$ correspond to the cases $\gamma(t) = t^{\gamma}$, $\lambda(t) = t^{\lambda}$, where γ, $\lambda \in \mathbb{R}$.

Properties analogous to those of the classes of type (A) and (W) are likewise valid for the classes of type $A(\infty)$ and $W(\infty)$. Note that these classes are non-empty for $\gamma < \lambda$: the power function $\varphi(t) = t^P$ satisfies condition $(Z_{\gamma})(\infty)$ for $P \geq \gamma$, $P > 0$ and condition $(Z_{\lambda})(\infty)$ for $P < \lambda$.

2. Some Applications of the Classes of Type (A) and (W)

2.1. Majorizing Classes. One of the most important applications of the classes of type (A) and (W) is that the pseudo-concave functions are used in the capacity of majorizing functions to characterize generalized Hölder spaces H^{φ}. This fact traces back to the works of S. M. Nikol'sky in which the functions $\omega(t)$ from the class of moduli of continuity Ω were used as majorizing functions.

In [1], the classes of majorizing functions Φ and Φ_k were introduced in connection with the necessities of the theory of function approximation and for obtaining generalized theorems related to Privalov's problems.

Different classes of majorizing functions have been introduced in the works due to Kh. Sh. Mukhtarov [3], S. G. Samko [2], N. G. Samko [14], [15] and also in [10]–[14], [17], etc.

2.2. On Fractional Integrals of Functions from the Classes A_λ^γ and W_λ^γ. Denote

$$\varphi_\lambda^* = t^\lambda \sup_{\tau \geq t} \frac{\varphi(\tau)}{\tau^\lambda} \quad \text{and} \quad \varphi^\gamma = t^\gamma \sup_{\tau \leq t} \frac{\varphi(\tau)}{\tau^\gamma}.$$

The functions $\varphi_\lambda^*(t)$ and $\varphi_*^\gamma(t)$ are the least majorizing functions of $\varphi(t)$ among all its majorants, which after dividing by t^λ and t^γ decrease and increase, respectively. From the definition of the functions φ_λ^* and φ^γ we get that $\varphi_\lambda^*(t) \sim \varphi_*^\gamma(t) \sim \varphi(t)$ for any function $\varphi(t) \in A_\lambda^\gamma$.

Consider the functions

$$(I^\gamma + \varphi)(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} \frac{\varphi_*^\gamma(t)}{t^\gamma}\, dt, \quad \gamma > 0$$

and

$$(J^\lambda + \varphi)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} \frac{\varphi_\lambda^*(t)}{t^\lambda}\, dt, \quad \lambda > 0$$

which are fractional integrals of $\frac{\varphi_*^\gamma(t)}{t^\gamma}$ and $\frac{\varphi_\lambda^*(t)}{t^\lambda}$ of order γ and λ, respectively. By simple transformations and reasoning it is not difficult to prove that $(I_+^\gamma \varphi)(x) \sim \varphi^*(x) \sim \varphi(x)$ for $\gamma > 0$ for $\varphi(x) \in A_\lambda^\gamma$ and $\varphi(x) \in W_\lambda^\gamma$; analogously, $(J_+^\lambda \varphi)(x) \sim \varphi^*(x) \sim \varphi(x)$ if $\varphi(x) \in A_\lambda^\gamma$ for $\gamma < \lambda < 1+\gamma$ and if $\varphi(x) \in W_\lambda^\gamma$ for $\gamma < \lambda \leq 1+\gamma$.

Thus the constructions I_+^λ and J_+^λ for the functions $\varphi(x)$ of the classes A_λ^γ and W_λ^γ are prescribed by the functions, equivalent to $\varphi(x)$, which have fractional derivatives up to orders γ and λ, respectively.

The function $(I_+^\lambda \varphi)(x)$ for $\lambda = k$ satisfies Corollary 2 from [9] (p. 159). Therefore by monotonicity of $\varphi^*(x)$ and $\frac{\varphi^*(x)}{x^\lambda}$ for $\lambda = k$ we have

$$\omega_k(T^k + \varphi, t) \sim (I^k + \varphi)(t) \sim \varphi^*(t) \sim \varphi(t).$$

2.3. Classes of Operators of Pseudo-Convex Type. Since the basic properties of functional classes A^α, A_β, A_β^α as well as of classes W^γ, W_λ, A_λ^γ are written by means of inequalities, we can introduce their natural generalizations to the case of classes of nonlinear operators of pseudo-concave–pseudo-convex at cones partially ordered spaces.

Definition 7. We shall say that the operator A belongs to the class V^γ, $\gamma \in \mathbb{R}$ (or to the class V_λ, $\lambda \in \mathbb{R}$), if the inequality

$$A(\alpha u) \leq \alpha^\gamma Au \ \ (\text{or} \ \ A(\alpha u) \geq \alpha^\lambda Au, \ \ \text{respectively}) \qquad (14)$$

is fulfilled for all $0 < \alpha < 1$ and the opposite inequality for $\alpha > 1$. Operators of the class V^γ is called pseudo-convex for $\gamma > 1$ and pseudo-concave for $\lambda < 1$ for V_λ.

Define a class of operators V_λ^γ as intersection of classes V^γ and V_λ:

$$V_\lambda^\gamma = V^\gamma \cap V_\lambda, \quad \gamma \le \lambda.$$

This class is empty for $\gamma > \lambda$.

Definition 8. We shall say that the operator A belongs to the class Λ^γ, $\gamma \in \mathbb{R}$ (or to the class Λ_λ, $\lambda \in \mathbb{R}$), if there exist $\mu_A > 0$ and $\nu_* > 0$ such that

$$A\alpha u \le \alpha^{\gamma+\mu_A} Au \ \ (\text{or} \ \ A\alpha u \ge \alpha^{\lambda-\nu_A} Au),$$

for $0 < \alpha < 1$ and an opposite inequality for $\alpha > 1$ (conf. (14)).

Define the class Λ_λ^γ as intersection of classes Λ^γ and Λ_λ:

$$\Lambda_\lambda^\gamma = \Lambda^\gamma \cap \Lambda_\lambda, \quad \gamma \le \lambda.$$

This class is empty for $\gamma \ge \lambda$.

We shall not dwell here on the properties of those classes. They can be easily obtained from the corresponding properties of the classes of type (A) and (W).

These classes are very effectively used for investigation of nonlinear operator equations of type $u = Au + f$, where A is the nonlinear operator from the classes of type V or Λ (see [4]–[6], [16], [20]).

2.4. Application to Nonlinear Integral Equations. Consider here two approaches to utilization of classes of type A, W for investigation of nonlinear Volterra integral equations of the type

$$u(x) = \int_0^x Q(x,t)\varphi\big[u(t)\big]\,dt + f(x) \tag{15}$$

where $Q(x,t)$ is a measurable nonnegative on $[0,\ell] \times [0,\ell]$ function and $[0,\ell] \subset \mathbb{R}^n$ is a parallelepiped in $\mathbb{R}^n$. Regarding the nonlinearity we assume that φ is the monotone function such that

$$\varphi(u) \in A_\lambda^\gamma \ \ \text{or} \ \ \varphi(u) \in W_\lambda^\gamma, \ \ -1 \le \gamma < \lambda \le 1 \tag{16}$$

On the basis of inequalities for $\varphi \in A_\lambda^\gamma$ we obtain a priori estimates for solutions $u(x)$ of the equation. Taking into account Property 2, we get

$$u(x) \ge \int_0^x Q_1(x,t)u^\gamma(t)\,dt + f(x), \quad u(x) \le \int_0^x Q_2(x,t)u^\gamma(t)\,dt + f(x),$$

where $Q_1(x,t) \le Q(x,t) \le Q_2(x,t)$. We apply the well-known theorems on integral inequalities with power nonlinearity whose theory is completely enough developed (see [18] and bibliography in [18]): solving equations

$$v(x) = \int_0^x Q_1(x,t)v^\lambda(t)\,dt + f(x) \ \ \text{and} \ \ \omega(x) = \int_0^x Q_2(x,t)\omega^\gamma(t)\,dt + f(x),$$

and taking into account (16), we find a priori estimates for the solution $u(x)$ of equation (15): $u(x) \geq v_0(x)$ and $u(x) \leq \omega_0(x)$ which define a cone segment $\langle v_0, \omega_0 \rangle$ containing a solution of the initial equation. Apply further the commonly known classical schemes from [4]–[6].

Consider the equation

$$u(x) = Au(x), \quad Au = \int\limits_{\Omega} Q(x,t)\varphi\big[u(t)\big]\,dt, \tag{17}$$

where Ω is the domain in $\mathbb{R}^n$, $Q(x,t)$ and $\varphi(u)$ have the properties: (a) $Q(x,t)$ is measurable and nonnegative on $\Omega \times \Omega$; (b) $\varphi(u)$ is monotone and $\varphi(u) \in A_\lambda^\gamma$, $-1 < \lambda < 1$.

Let K be a cone of positive in Ω functions and the operator A be defined on K. Let $g(x) \in K$. Denote by K_g a constituent of the cone K from functions $u(x)$, equivalent to the function $g(x)$.

Theorem 13. *Let conditions* (a) *and* (b) *be fulfilled and let there exist a function* $g(x) \in K$ *such that* $A_g \sim g$. *Then in* K_g *there exists a unique solution* $u_*(x)$ *of equation* (17) *and*

$$u_*(x) = \lim_{n\to\infty} A^n g(x).$$

It can be easily shown that in the case of convolutional kernel $Q(x,t) = K(x-t)$ for almost weak anti-synchronous (see [19], [20]) $K(x)$ and $u(x)$ the function $g(x)$ can be constructed explicitly with respect to the kernel $K(x)$ and nonlinearity of φ. Let we have the equation

$$Au = \int\limits_0^x K(x-t)\varphi\big[u(t)\big]\,dt.$$

Denote $x \in (x_1,\dots,x_n) \in [0,\ell] = [0,\ell_1] \times \cdots \times [0,\ell_n]$, $K(x) > 0$, $x \in [0,\ell]$, $\varphi \in A_0\lambda^*$, $0 < \lambda \leq 1$, $\ell = (\ell_1,\dots,\ell_n)$, $\ell_i > 0$, $i = \overline{1,n}$.

This function is monotone, since there exist $G^{-1}(u)$, where $\frac{G^{-1}(u)}{u}$ increases and $\frac{G^{-1}(u)}{u^{\frac{1}{1-\lambda}}}$ decreases.

Suppose

$$g(x) = G^{-1}\Big(\int\limits_0^x K(t)\,dt\Big).$$

Then under the additional assumption

$$\int\limits_0^x K(t)\Big(\int\limits_0^t K(\tau)\Big)^{\frac{1}{1-\lambda}} dt \geq C\Big(\int\limits_0^x K(t)\,dt\Big)^{\frac{1}{1-\nu}}$$

(appearing for $n > 1$), we find that $A_g \sim g$ (in this connection see also [4]–[6]).

Note that in terms of the classes of type (A) and (W) on every interim step of transformations we have an exhaustive information on the smoothness of an obtainable expression, including also of coefficient values of almost monotonicity of these expressions.

2.5. Multiplicative Inequalities of Kolmogorov Type. The following lemma is actually used in proving interpolation theorems for generalized Hölder spaces H^{φ}.

Lemma 14. *Let* $\frac{\gamma_1(t)}{\gamma_2(t)}, \frac{\gamma_2(t)}{\gamma_3(t)} \in W_0$ *and* $\frac{\lambda_1(t)}{\lambda_2(t)}, \frac{\lambda_2(t)}{\lambda_3(t)} \in W_0$. *Then for every functions* $\varphi_1(t) \in W^{\gamma_1(t)}_{\lambda_1(t)}$ *and* $\varphi_3(t) \in W^{\gamma_3(t)}_{\lambda_3(t)}$ *there exists the function* $\varphi_2(t) \in W^{\gamma_2(t)}_{\lambda_2(t)}$ *such that*

$$\varphi_2(t) = \varphi_1^{\nu}(t) \cdot \varphi_3^{\nu}(t), \tag{18}$$

where $0 < \nu = \ln \frac{\gamma_2(t)}{\gamma_3(t)} \big/ \ln \frac{\gamma_1(t)}{\gamma_3(t)} = \ln \frac{\lambda_2(t)}{\lambda_3(t)} \big/ \ln \frac{\lambda_1(t)}{\lambda_3(t)} < 1$, *and vice versa, for every function* $\varphi_2(t) \in W^{\gamma_2(t)}_{\lambda_2(t)}$ *there exist* $\varphi_1(t) \in W^{\gamma_1(t)}_{\lambda_1(t)}$ *and* $\varphi_3(t) \in W^{\gamma_3(t)}_{\lambda_3(t)}$ *such that equality* (18) *is fulfilled.*

Proof. We have

$$\varphi_2(t) = \varphi_1^{\nu}(t) \cdot \varphi_3^{1-\nu} \in W^{\gamma_1^{\nu}(t)\cdot\gamma_3^{1-\nu}(t)}_{\lambda_1^{\nu}(t)\cdot\lambda_3^{1-\nu}(t)} = W^{\gamma_2(t)}_{\lambda_2(t)}. \qquad \square$$

In the case of classes $W^{\gamma(t)}$ or $W_{\lambda(t)}$ the proof is similar.

Theorem 15. *Let* H^{φ_i}, $i = \overline{1,3}$ *be generalized Hölder spaces with naturally introduced norms* $\|f\|_{H^{\varphi_i}}$, $i = \overline{1,3}$, *where* $\varphi_i(t) \in W^{\gamma_i}_{\lambda_i}$, $i = \overline{1,3}$ *satisfy the conditions of Lemma* 14. *Then the inequality of Kolmogorov type is valid:*

$$\|f\|_{H^{\varphi_2}} \le C \cdot \|f\|^{\nu}_{H^{\varphi_1}} \cdot \|f\|^{1-\nu}_{H^{\varphi_3}},$$

where ν *is defined just in the same way as in Lemma* 14.

On the basis of Lemma 14 and Theorem 15 we can prove in the standard manner that generalized Hölder spaces with natural norms form a scale of Banach spaces (see also [3], [7]).

2.6. Improvement of Some Classical Results. Application of classes W^{γ}_{λ} allows one to improve the well-known results for moduli of continuity (see, for e.g., [8], Theorem 348, pp. 417–421).

Theorem 16. *Let the integral smoothness moduli* $\omega_k(f^{(r)}, \delta)_p$ *and* $\omega_k(f, \delta)_p$, $1 \le p \le \infty$, $k = 1, 2, \ldots$, *satisfy the conditions*

$$\int_0^{\delta} \frac{\omega_k(f^{(r)}, t)_p}{t}\, dt \le B_0 \omega_k(f^{(r)}, \delta)_p \tag{19}$$

and

$$\delta^k \int\limits_{\delta}^{\ell} \frac{\omega_k(f,t)_p}{t^{1+k}}\,dt \le B_k \omega_k(f,\delta)_p. \tag{20}$$

Then there exist $0<\mu<1$ *and* $0<\nu<k$ *such that the inequalities*

$$\omega_k(f^{(r)},\alpha\delta)_p \le C_0 \cdot \omega_k(f^{(r)},\delta)\cdot \alpha^\mu$$

and

$$\omega_k(f,\delta)_p \le C_k \cdot \omega_k(f,\alpha\delta)\cdot \frac{1}{\alpha^k}\cdot \alpha^\nu,$$

uniformly with respect to all $0<\alpha<1/2$*,* $0<\delta<1/2$ *are, respectively, valid.*

Proof. Relying on the theorems on the equivalence, it follows from (19) and (20) that $\omega_k(f^{(r)},\delta)_p \in W^0 = \Phi^0$ and $\omega_k(f,\delta)_p \in W_k = \Phi_k$. By conditions (S_γ) and (S_λ) we obtain the existence of the required values μ, $\nu>0$; they can be calculated explicitly by the values B_0 and B_k.

Note that in [8] these statements have been obtained with the factor $\frac{1}{|\ln\alpha|}$ instead of α^μ and α^ν. □

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