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MACKEY TOPOLOGIES OF ORLICZ-BOCHNER SPACES

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Abstract. For a real Banach space X let $L^\varphi(X)$ be an Orlicz-Bochner spaces defined by an Orlicz function φ taking only finite values (not necessarily convex) over a σ-finite atomless measure space provided with its complete metrizable topology $\mathcal{T}_\varphi(X)$. In the paper it is shown that the Mackey topology $\tau_{L^\varphi(X)}$ of $(L^\varphi(X), \mathcal{T}_\varphi(X))$ coincides with the supremum of the topology $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$ ($\overline{\varphi} =$ the convex minorant of φ) and the topology $\pi_\varphi(X)$ of the Minkowski functional of the Orlicz class $L_0^\varphi(X)$. Necessary and sufficient conditions for $\tau_{L^\varphi(X)}$ to be identical with $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$ are given in terms of φ and $\overline{\varphi}$.

1. Introduction and preliminaries.

Let us recall that the Mackey topology of a topological vector space (L, ξ) is the finest locally convex topology τ_L that produces the same continuous linear functionals as ξ. It is known that if ξ is a metrizable topology then τ_L is the finest locally convex topology on L that is weaker than ξ and has a base at zero consisting of convex hulls of neighbourhoods of zero for ξ (see [18]).

N.J. Kalton [9] showed that if $(\ell^\varphi, \mathcal{T}_\varphi)$ is a separable Orlicz sequence space (i.e. φ satisfies the Δ_2-condition at zero) then its Mackey topology τ_φ coincides with the topology $\mathcal{T}_{\hat{\varphi}|\ell^\varphi}$ induced from $(\ell^{\hat{\varphi}}, \mathcal{T}_{\hat{\varphi}})$, where $\hat{\varphi}$ is the convex minorant of φ in a neighbourhood of zero. L. Drewnowski and M.

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Nawrocki [5] proved that the Mackey topology τ_φ of an arbitrary Orlicz sequence spaces coincides with the supremum of $\mathcal{T}_{\hat{\varphi}|\ell^\varphi}$ and π_φ, where π_φ is the topology of the Minkowski functional of an Orlicz class ℓ_0^φ. N.J. Kalton [10] and L. Drewnowski [6] investigated the Mackey topology on Orlicz spaces and Musielak-Orlicz spaces defined by finite valued Orlicz functions over an atomless measure space. These results allow us to use methods of the theory of locally convex vector spaces to develop the duality theory of non-locally convex Orlicz spaces (see [11], [15], [16]). In particular, the general form of continuous linear functionals on non-locally convex Orlicz spaces was found (see [15], [16]).

In this paper Orlicz-Bochner spaces $L^\varphi(X)$ defined by a finite valued Orlicz function (not necessarily convex) over a σ-finite atomless measure space are considered. These spaces are a natural generalization of Lebesgue-Bochner spaces $L^p(X)$ (see [3], [8]). By making use of [6, Corollary 2] and following the idea of the proof of [5, Theorem 5.3] we obtain a description of the Mackey topology $\tau_{L^\varphi(X)}$ of $(L^\varphi(X), \mathcal{T}_\varphi(X))$. In [17] we apply this result to characterize the topological dual of $L^\varphi(X)$.

For terminology concerning Riesz spaces we refer to [1]. Throughout the paper let (Ω, Σ, μ) be a σ-finite atomless measure space and let L^0 stand for the corresponding space of equivalence classes of all Σ-measurable real valued functions defined and finite μ-a.e. For a subset A of Ω let χ_A stand for its characteristic function. Let $\mathbb{N}$ stand for the set of all natural numbers.

Let $(X, \|\cdot\|_X)$ be a real Banach space, and let S_X and B_X denote the unit sphere and the unit ball in X resp. By $L^0(X)$ we will denote the linear space of equivalence classes of all strongly Σ-measurable functions $f\colon \Omega \to X$. For a function $f \in L^0(X)$ let us put $\tilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Now we recall some terminology concerning Orlicz spaces and Orlicz-Bochner spaces (see [3], [8], [12], [13], [14], [19]).

By an *Orlicz function* we mean here a function $\varphi\colon [0,\infty) \to [0,\infty)$ that is non-decreasing, left continuous, continuous at 0 with $\varphi(0) = 0$. An Orlicz function φ is said to be *strict* if it is not identically equal to zero.

For an Orlicz function φ by $\overline{\varphi}$ we will denote its *convex minorant*, i.e., $\overline{\varphi}$ is the largest convex Orlicz function that is smaller than φ on $[0,\infty)$. Clearly $\overline{\varphi}$ is strict iff $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} > 0$.

Let φ be an Orlicz function. For each $u \in L^0$ we define

$$m_\varphi(u) = \int_\Omega \varphi(|u(\omega)|)d\mu.$$

The Orlicz space L^φ defined by φ is an ideal of L^0 defined by

$$L^\varphi = \{u \in L^0\colon\ m_\varphi(\lambda u) < \infty \quad \text{for some} \quad \lambda > 0\}$$

and endowed with the complete semimetrizable topology $\mathcal{T}_\varphi$ of the Riesz pseudonorm $|u|_\varphi = \inf\{\lambda > 0\colon\ m_\varphi(u/\lambda) \leqslant \lambda\}$. $\mathcal{T}_\varphi$ is a Hausdorff topology iff φ is strict. The functional restricted to L^φ is a modular (see [12], [13], [14]). Moreover, if φ is a convex Orlicz function then $\mathcal{T}_\varphi$ can be generated by the seminorm $\|u\|_\varphi = \inf\{\lambda > 0\colon\ m_\varphi(u/\lambda) \leqslant 1\}$.

Let $E^\varphi = \{u \in L^0\colon\ m_\varphi(\lambda u) < \infty$ for all $\lambda > 0\}$. Then $E^\varphi = (L^\varphi)_a$ (= the ideal of absolutely continuous elements of L^φ) and $L^\varphi = E^\varphi$ iff φ satisfies the Δ_2-condition (i.e. $\limsup \frac{\varphi(2t)}{\varphi(t)} < \infty$ as $t \to 0$ and $t \to \infty$).

Let

$$L^\varphi(X) = \{f \in L^0(X)\colon\ \widetilde{f} \in L^\varphi\} \quad \text{and} \quad E^\varphi(X) = \{f \in L^0(X)\colon\ \widetilde{f} \in E^\varphi\}.$$

The space $L^\varphi(X)$ is called an *Orlicz-Bochner* space and can be endowed with a complete semimetrizable topology $\mathcal{T}_\varphi(X)$ of the F-pseudonorm $|f|_{L^\varphi(X)} = |\widetilde{f}|_\varphi$ for $f \in L^\varphi(X)$. If φ is a convex Orlicz function then $\mathcal{T}_\varphi(X)$ can be generated by the seminorm $\|f\|_{L^\varphi(X)} = \|\widetilde{f}\|_\varphi$.

2. The Mackey topology of $E^\varphi(X)$.

In this section we will assume that φ is a strict Orlicz function. We shall show that the Mackey topology $\tau_{E^\varphi(X)}$ of $(E^\varphi(X), \mathcal{T}_\varphi(X)|_{E^\varphi(X)})$ coincides with the seminormable topology $\mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$ induced from $(L^{\overline{\varphi}}(X), \mathcal{T}_{\overline{\varphi}}(X))$. For this purpose we shall use the following description of the Mackey topology τ_{E^φ} of $(E^\varphi, \mathcal{T}_\varphi|_{E^\varphi})$ (cf. [6, Corollary 2]).

Theorem 2.1. *The Mackey topology τ_{E^φ} of $(E^\varphi, \mathcal{T}_\varphi|_{E^\varphi})$ coincides with the seminormable topology $\mathcal{T}_{\overline{\varphi}}|_{E^\varphi}$ induced from $(L^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}})$, i.e., $\tau_{E^\varphi} = \mathcal{T}_{\overline{\varphi}}|_{E^\varphi}$. Moreover, τ_{E^φ} is normable iff $\liminf\limits_{t\to\infty} \frac{\varphi(t)}{t} > 0$.*

We shall need two definitions (see [7]).

A pseudonorm ρ on $L^\varphi(X)$ is said to be *solid* if $\rho(f_1) \leqslant \rho(f_2)$ whenever $f_1, f_2 \in L^\varphi(X)$ with $\widetilde{f_1} \leqslant \widetilde{f_2}$.

A subset H of $L^\varphi(X)$ is said to be solid if $\widetilde{f_1} \leqslant \widetilde{f_2}$ with $f_1 \in L^\varphi(X)$, $f_2 \in H$ imply $f_1 \in H$.

For $\varepsilon > 0$ let

$$B_\varphi(\varepsilon) = \{u \in E^\varphi\colon\ |u|_\varphi \leqslant \varepsilon\}, \quad W_\varphi(\varepsilon) = \operatorname{conv} B_\varphi(\varepsilon),$$
$$B_\varphi(X,\varepsilon) = \{f \in E^\varphi(X)\colon\ |f|_{L^\varphi(X)} \leqslant \varepsilon\}, \quad W_\varphi(X,\varepsilon) = \operatorname{conv} B_\varphi(X,\varepsilon).$$

In view of [1, Theorem 1.3] $W_\varphi(\varepsilon)$ is a solid subset of E^φ. Moreover, by [7, Theorem 1.2] $W_\varphi(X,\varepsilon)$ is a solid subset of $E^\varphi(X)$. Since $\mathcal{T}_\varphi|_{E^\varphi}$ is a metrizable topology, the family $\{W_\varphi(m^{-1}) : m \in \mathbb{N}\}$ forms a local base at zero for the Mackey topology τ_{E^φ}. Similarly, the family $\{W_\varphi(X, m^{-1}) : m \in \mathbb{N}\}$ is a local base at zero for the Mackey topology $\tau_{E^\varphi(X)}$.

For $m \in \mathbb{N}$ let us set

$$p_m(u) = \inf\{\lambda > 0\colon\ u \in \lambda W_\varphi(m^{-1})\} \quad \text{for} \quad u \in E^\varphi,$$
$$\rho_m(f) = \inf\{\lambda > 0\colon\ f \in \lambda W_\varphi(X, m^{-1})\} \quad \text{for} \quad f \in E^\varphi(X).$$

It is clear that p_m and ρ_m $(m = 1, 2, \dots)$ are solid seminorms on E^φ and $E^\varphi(X)$ resp. Moreover, the Mackey topologies τ_{E^φ} and $\tau_{E^\varphi(X)}$ are generated by the familes $\{p_m\colon\ m \in \mathbb{N}\}$ and $\{\rho_m\colon\ m \in \mathbb{N}\}$ resp.

Given a solid (= Riesz) seminorm p on E^φ let us set

$$\overline{p}\,(f) = p(\widetilde{f}) \quad \text{for all} \quad f \in E^\varphi(X).$$

It is easy to check that $\overline{p}$ is a solid seminorm on $E^\varphi(X)$.

For $u \in E^\varphi$ let $\overline{u}\,(\omega) = u(\omega)x$ for some $x \in S_X$ and all $\omega \in \Omega$. Then $\overline{u}, \in L^0(X)$ and $\|\,\overline{u}\,(\omega)\|_X = |u(\omega)|$ for $\omega \in \Omega$, so $\overline{u} \in E^\varphi(X)$. Given a solid seminorm ρ on $E^\varphi(X)$ let us set

$$\widetilde{\rho}\,(u) = \rho(\overline{u}) \quad \text{for all} \quad u \in E^\varphi.$$

Note that $\widetilde{\rho}$ is a solid seminorm on E^φ (see [7, §3]).

We shall need the following lemma.

Lemma 2.2. *Let $u \in E^\varphi$ and $\varepsilon > 0$. Then $u \in W_\varphi(\varepsilon)$ iff $\overline{u} \in W_\varphi(X, \varepsilon)$.*

Proof. (i) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (i). Let $\overline{u} \in W_\varphi(X, \varepsilon)$, i.e., $\overline{u} = \sum_{i=1}^n \alpha_i f_i$, where $|f_i|_{L^\varphi(X)} = |\widetilde{f_i}|_\varphi \leqslant \varepsilon$ and $\alpha_i \geqslant (i = 1, 2, \dots, n)$ with $\sum_{i=1}^n \alpha_i = 1$. Then

$$\widetilde{\overline{u}}\,(\omega) = \Big\|\sum_{i=1}^n \alpha_i f_i(\omega)\Big\|_X \leqslant \sum_{i=1}^n \alpha_i \|f_i(\omega)\|_X = \Big(\sum_{i=1}^n \alpha_i \widetilde{f_i}\Big)(\omega).$$

Since $\sum_{i=1}^n \alpha_i \widetilde{f_i} \in W_\varphi(\varepsilon)$, we get $\widetilde{\overline{u}}, \in W_\varphi(\varepsilon)$, because $W_\varphi(\varepsilon)$ is a solid subset of L^φ. But $\widetilde{\overline{u}} = |u|$, so $u \in W_\varphi(\varepsilon)$, as desired. ■

As an application of Lemma 2.2 we have:

Theorem 2.3. *For $m \in \mathbb{N}$ we have*

$$\widetilde{\rho}_m\,(u) = p_m(u) \ \textit{for all}\ u \in E^\varphi \quad \textit{and} \quad \overline{p}_m\,(f) = \rho_m(f) \ \textit{for all}\ f \in E^\varphi(X).$$

Proof. For $u \in E^\varphi$ by Lemma 2.2 we get

$$\begin{aligned}\widetilde{\rho}_m\,(u) = \rho_m(\overline{u}) &= \inf\left\{\lambda > 0\colon\ \overline{u} \in \lambda W_\varphi\left(X, m^{-1}\right)\right\}\\ &= \inf\left\{\lambda > 0\colon\ u \in \lambda W_\varphi\left(m^{-1}\right)\right\} = p_m(u).\end{aligned}$$

For $f \in E^\varphi(X)$ we have $\|\,\overline{\widetilde{f}}\,(\omega)\|_X = \widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$ and by the solideness of the sets $W_\varphi(X, \varepsilon)$ we get

$$\begin{aligned}\overline{p}_m\,(f) = p_m(\widetilde{f}) &= \inf\left\{\lambda > 0\colon\ \widetilde{f} \in W_\varphi\left(m^{-1}\right)\right\}\\ &= \inf\left\{\lambda > 0\colon\ \overline{\widetilde{f}} \in \lambda W_\varphi\left(X, m^{-1}\right)\right\}\\ &= \inf\left\{\lambda > 0\colon\ f \in \lambda W_\varphi\left(X, m^{-1}\right)\right\} = \rho_m(f). \quad ■\end{aligned}$$

Assume that τ is a locally convex topology on $E^\varphi(X)$ generated by a family $\{\rho_\alpha\colon \alpha \in \{\alpha\}\}$ of solid seminorms on $E^\varphi(X)$. We will denote by $\widetilde{\tau}$ the topology on E^φ generated by the family $\{\widetilde{\rho}_\alpha\colon \alpha \in \{\alpha\}\}$.

In turn, let ξ be a locally convex topology on E^φ generated by a family $\{p_\alpha\colon \alpha \in \{\alpha\}\}$ of solid seminorms on E^φ. We will denote by $\overline{\xi}$ the topology on $E^\varphi(X)$ generated by the family $\{\overline{p}_\alpha\colon \alpha \in \{\alpha\}\}$.

Now we are in position to prove our desired result.

Theorem 2.4. *The identity* $\tau_{E^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$ *holds and* $\tau_{E^\varphi(X)}$ *is normable iff* $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$.

Proof. We know that the Mackey topology $\tau_{E^\varphi(X)}$ is generated by the family $\{\rho_m\colon m \in \mathbb{N}\}$. Hence by Theorem 2.3 the topology $\widetilde{\tau}_{E^\varphi(X)}$ is generated by the family $\{p_m\colon m \in \mathbb{N}\}$, so $\widetilde{\tau}_{E^\varphi(X)}= \tau_{E^\varphi}$. In view of [7, Theorem 3.2] the identity $\overline{\widetilde{\tau}}_{E^\varphi(X)}= \tau_{E^\varphi(X)}$ holds, so $\tau_{E^\varphi(X)} =\overline{\tau}_{E^\varphi}$.

On the other hand, by Theorem 2.1 the Mackey topology τ_{E^φ} is generated by $\cdot_{\overline{\varphi}}$, so $\overline{\tau}_{E^\varphi}$ is generated by $|\cdot|_{L^{\overline{\varphi}}(X)}$. Hence the identity $\tau_{E^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$ holds, as desired. ■

As an application of Theorem 2.4 we have the following results.

Corollary 2.5. *The following statements are equivalent:*
(i) *The space* $(L^\varphi(X), \mathcal{T}_\varphi(X))$ *is locally convex.*
(ii) *The space* $(E^\varphi(X), \mathcal{T}_\varphi(X)|_{E^\varphi(X)})$ *is locally convex.*
(iii) φ *is equivalent to* $\overline{\varphi}$.

Proof. (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (iii) We have $\tau_{E^\varphi(X)} = \mathcal{T}_\varphi(X)|_{E^\varphi(X)}$. Hence by Theorem 2.4 $\mathcal{T}_\varphi(X)|_{E^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$, so $\mathcal{T}_\varphi|_{E^\varphi} = \mathcal{T}_\varphi|_{E^{\overline{\varphi}}}$. It follows that φ is equivalent to $\overline{\varphi}$. ■

Corollary 2.6. *The completion of* $(E^\varphi(X), \tau_{E^\varphi(X)})$ *is equal to* $(E^{\overline{\varphi}}(X), \mathcal{T}_{\overline{\varphi}}(X)|_{E^{\overline{\varphi}}(X)})$.

3. The Mackey topology of Orlicz-Bochner spaces.

Denote by $\tau_{L^\varphi(X)}$ the Mackey topology of $(L^\varphi(X), \mathcal{T}_\varphi(X))$. It is known that $\tau_{L^\varphi(X)}$ is the finest locally convex topology on $L^\varphi(X)$ that is weaker than $\mathcal{T}_\varphi(X)$. In this section following the idea of [5] we obtain an important description of $\tau_{L^\varphi(X)}$.

Lemma 3.1. *The following identities hold:*

$$\tau_{L^\varphi(X)}|_{E^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)} = \tau_{E^\varphi(X)}.$$

Proof. Clearly $\mathcal{T}_{\overline{\varphi}}(X) \subset \tau_{L^\varphi(X)}$. Since $\tau_{L^\varphi(X)}|_{E^\varphi(X)} \subset \mathcal{T}_\varphi(X)|_{E^\varphi(X)}$ we get k $\tau_{L^\varphi(X)}|_{E^\varphi(X)} \subset \tau_{E^\varphi(X)}$. Thus by Theorem 2.4 the proof is complete.

The Orlicz class $L_0^\varphi(X) = \{f \in L^\varphi(X)\colon \int_\Omega \varphi(\widetilde{f})d\mu < \infty\}$ is an absolutely convex absorbing subset of $L^\varphi(X)$ and let K_φ stand for its Minkowski functional, i.e.,

$$K_\varphi(f) = \inf\{\lambda > 0\colon \int_\Omega \varphi(\widetilde{f}/\lambda)d\mu < \infty\}.$$

for $f \in L^\varphi(X)$. Note that K_φ is a solid seminorm on $L^\varphi(X)$ and $\ker K_\varphi = E^\varphi(X)$. Moreover $K_\varphi(f) \leqslant f_{L^\varphi(X)}$ for $f \in L^\varphi(X)$. Since supp $E^\varphi = \Omega$ there exists a sequence (Ω_n) in Σ such that $\Omega_n \uparrow \Omega$ and $\chi_{\Omega_n} \in E^\varphi$ (see [20, Theorem 86.2]). Given $f \in L^\varphi(X)$ let us put for $n = 1, 2, \ldots$

$$f^{(n)}(\omega) = \begin{cases} f(\omega) & \text{if } \widetilde{f}(\omega) \leqslant n \text{ and } \omega \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

and

$$\widetilde{f}^{(n)}(\omega) = \begin{cases} \widetilde{f}(\omega) & \text{if } \widetilde{f}(\omega) \leqslant n \text{ and } \omega \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly $f^{(n)} \in E^\varphi(X)$ for $n = 1, 2, \ldots$ and

$$\widetilde{f - f^{(n)}}(\omega) = \widetilde{f}(\omega) - \widetilde{f}^{(n)}(\omega) = \begin{cases} 0 & \text{if } \widetilde{f}(\omega) \leqslant n \text{ and } \omega \in \Omega_n, \\ \widetilde{f}(\omega) & \text{elsewhere.} \end{cases} \qquad \blacksquare$$

Lemma 3.2. *For $f \in L^\varphi(X)$ we have*

$$K_\varphi(f) = \inf\{|f - h|_{L^\varphi(X)}\colon h \in E^\varphi(X)\} = \lim_n |f - f^{(n)}|_{L^\varphi(X)}.$$

Proof. We have $|f - f^{(n)}|_{L^\varphi(X)} = \widetilde{f} - \widetilde{f}^{(n)}{}_\varphi$. Choose $\lambda > 0$ such that $\int_\Omega \varphi(\widetilde{f}/\lambda) < \infty$. Since $\widetilde{f}(\omega) - \widetilde{f}^{(n)}(\omega) \downarrow_n 0$ μ-a.e., there exists $n_0 \in \mathbb{N}$ such that $\int_\Omega \varphi(|\widetilde{f}(\omega) - \widetilde{f}^{(n)}(\omega)|/\lambda)d\mu \leqslant \lambda$ for $n \geqslant n_0$, so $|f - f^{(n)}|_{L^\varphi(X)} \leqslant \lambda$. Hence $\inf\{|f - h|_{L^\varphi(X)}\colon h \in E^\varphi(X)\} \leqslant \lim_n |f - f^{(n)}|_{L^\varphi(X)} \leqslant K_\varphi(f)$.

On the other hand, let $h \in E^\varphi(X) = \ker K_\varphi$. Then $K_\varphi(f) \leqslant K_\varphi(f - h) \leqslant |f - h|_{L^\varphi(X)}$, so $K_\varphi(f) \leqslant \inf\{|f - h|_{L^\varphi(X)}\colon h \in E^\varphi(X)\}$.

Thus the proof is complete. $\blacksquare$

The quotient topology $\mathcal{T}_\varphi(X)/E^\varphi(X)$ on $L^\varphi(X)/E^\varphi(X)$ is generated by the F-norm $[f]_{L^\varphi(X)} = \inf\{|f - h|_{L^\varphi(X)}\colon h \in E^\varphi(X)\}$. Denote by $\pi_\varphi(X)$ the topology on $L^\varphi(X)$ of the seminorm K_φ. We know that the quotient topology $\pi_\varphi(X)/E^\varphi(X)$ on $L^\varphi(X)/E^\varphi(X)$ is generated by the seminorm $K_\varphi([f]) = \inf\{K_\varphi(f - h)\colon h \in E^\varphi(X)\}$. Since $K_\varphi([f]) = K_\varphi(f)$ for $f \in L^\varphi(X)$, by Lemma 3.2 we get

Corollary 3.3. *The identity $\mathcal{T}_\varphi(X)/E^\varphi(X) = \pi_\varphi(X)/E^\varphi(X)$ holds.*

We are ready to state our main result.

Theorem 3.4. *The Mackey topology $\tau_{L^\varphi(X)}$ coincides with the supremum of $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$ and $\pi_\varphi(X)$, i.e., $\tau_{L^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)} \vee \pi_\varphi(X)$.*

Proof. Denote by $\xi = \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)} \vee \pi_\varphi(X)$. Clearly $\xi \subset \tau_{L^\varphi(X)}$, because $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)} \subset \mathcal{T}_\varphi(X)$ and $\pi_\varphi(X) \subset \mathcal{T}_\varphi(X)$. Since $\xi|_{E^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)} \vee \pi_\varphi(X)|_{E^\varphi(X)}$ and $\pi_\varphi(X)|_{E^\varphi(X)} \subset \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)}$, by Lemma 3.1 we get

$$\xi|_{E^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{E^\varphi(X)} = \tau_{L^\varphi(X)}|_{E^\varphi(X)}. \tag{1}$$

We have $\pi_\varphi(X) \subset \xi$, $\xi \subset \tau_{L^\varphi(X)}$ and $\tau_{L^\varphi(X)} \subset \mathcal{T}_\varphi(X)$. Hence $\pi_\varphi(X)/E^\varphi(X) \subset \xi/E^\varphi(X) \subset \tau_{L^\varphi(X)}/E^\varphi(X) \subset \mathcal{T}_\varphi(X)/E^\varphi(X)$. By Corollary 3.3 the quotient topologies $\pi_\varphi(X)/E^\varphi(X)$ and $\mathcal{T}_\varphi(X)/E^\varphi(X)$ on $L^\varphi(X)/E^\varphi(X)$ coincide, so

$$\xi/E^\varphi(X) = \tau_{L^\varphi(X)}/E^\varphi(X). \tag{2}$$

In view of (1) and (2) by [4, Lemma 2.1], $\xi = \tau_{L^\varphi(X)}$, as desired. ■

Corollary 3.5. (i) *If $\liminf_{t\to\infty} \frac{\varphi(t)}{t} = 0$, then $\tau_{L^\varphi(X)} = \pi_\varphi(X)$.*

(ii) *If φ satisfies the Δ_2-condition, then $\tau_{L^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$.*

The next theorem present necessary and sufficient conditions for the Mackey topology $\tau_{L^\varphi(X)}$ to be identical with $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$ (cf. [5, Corollary 5.5]).

Theorem 3.6. *The following statements are equivalent:*

(i) $\tau_{L^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$.

(ii) $L^\varphi(X) \cap E^{\overline{\varphi}}(X) \subset E^\varphi(X)$.

(iii) $L_0^\varphi(X) \cap \frac{1}{b} L_0^{\overline{\varphi}}(X) \subset \frac{1}{2} L_0^\varphi(X)$ *for some* $b > 0$.

(iv) $K_\varphi(f) \leqslant \frac{1}{b} K_{\overline{\varphi}}(f)$ *for some* $b > 0$ *and all* $f \in L^\varphi(X)$.

(v) *There exist* $a > 0$, $b > 0$ *such that*

$$\varphi(2t) \leqslant a \max(\varphi(t), \overline{\varphi}(bt)) \quad \textit{for all} \quad t \geqslant 0.$$

Proof. (i) ⇒ (ii) Assume that $\tau_{L^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$. In view of Theorem 3.4 we conclude that $\pi_\varphi(X) \subset \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$. To prove that $L^\varphi(X) \cap E^{\overline{\varphi}}(X) \subset E^\varphi(X)$ let $f \in L^\varphi(X) \cap E^{\overline{\varphi}}(X)$. Since $f \in E^{\overline{\varphi}}(X)$ we have that $\|f - f^{(n)}\|_{L^{\overline{\varphi}}(X)} \to 0$, so $K_\varphi(f - f^{(n)}) \to 0$. It follows that $K_\varphi(f) = 0$, because $K_\varphi(f) \leqslant K_\varphi(f - f^{(n)}) + K_\varphi(f^{(n)})$ and $f^{(n)} \in E^\varphi(X) = \ker K_\varphi$. Hence $f \in E^\varphi(X)$.

(ii) ⇒ (iii) Suppose that (iii) is false, i.e., $L_0^\varphi(X) \cap \frac{1}{n} L_0^{\overline{\varphi}}(X) \not\subset \frac{1}{2} L_0^\varphi(X)$ for $n = 1, 2, \ldots$. Hence there exists a sequence (f_n) in $L^0(X)$ such that

$$m_\varphi(\widetilde{f}_n) < \infty, \quad m_{\overline{\varphi}}(n\widetilde{f}_n) < \infty, \quad m_\varphi(2\widetilde{f}_n) = \infty.$$

For $n = 1, 2, \ldots$ let us put

$$\widetilde{f}_n^{(k)}(\omega) = \begin{cases} \widetilde{f}_n(\omega) & \text{if} \quad \omega \in \Omega_k, \quad \widetilde{f}_n(\omega) \leqslant k, \\ 0 & \text{elsewhere,} \end{cases}$$

for $k = 1, 2, \ldots$. Then there exists a strictly increasing sequence (k_n) in $\mathbb{N}$ such that

$$m_\varphi(\widetilde{f}_n - \widetilde{f}_n^{(k_n)}) < 2^{-n}, \quad m_{\overline{\varphi}}(n(\widetilde{f}_n - \widetilde{f}_n^{(k_n)})) < 2^{-n}, \quad m_\varphi(2(\widetilde{f}_n - \widetilde{f}_n^{(k_n)})) > n.$$

Let $u_n = \widetilde{f}_n - \widetilde{f}_n^{(k_n)}$ for $n = 1, 2, \ldots$. Then according to [13] there exists $u \in L^\varphi$ such that $u = \sup_n u_n$ and $m_\varphi(u) \leqslant \sum_{n=1}^\infty m_\varphi(u_n) < \infty$, $m_{\overline{\varphi}}(ru) \leqslant \sum_{n=1}^\infty m_{\overline{\varphi}}(ru_n) < \infty$ for all $r > 1$ and $m_\varphi(2u) \geqslant m_\varphi(2u_n) \geqslant n$, so $m_\varphi(2u) = \infty$. Putting $f(\omega) = u(\omega)x_0$ for some $x_0 \in S_X$ and all $\omega \in \Omega$ we have $f \in L^\varphi(X) \cap E^{\overline{\varphi}}(X)$ and $f \notin E^\varphi(X)$, so that (ii) does not hold.

(iii) $\Rightarrow$ (iv) It is easy to see that (iii) implies that $L^\varphi(X) \cap L_0^{\overline{\varphi}}(X) \subset bL_0^\varphi(X)$ holds. It follows that $K_\varphi(f) \leqslant \frac{1}{b} K_{\overline{\varphi}}(f)$ for all $f \in L^\varphi(X)$.

(iv) $\Rightarrow$ (i) If (iv) holds, then $\pi_\varphi(X) \subset \pi_{\overline{\varphi}}(X)|_{L^\varphi(X)}$. Since $\pi_{\overline{\varphi}}(X) \subset \mathcal{T}_{\overline{\varphi}}(X)$, in view of Theorem 3.4 $\tau_{L^\varphi(X)} = \mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$.

(v) $\Leftrightarrow$ (i) It follows from the general properties of Orlicz spaces. ■

Remark. (i) In [11] one can find an example of an Orlicz function φ which does not satisfy the Δ_2-condition and $\overline{\varphi}(t) = t$ for $t \geqslant 0$. It follows that φ does not satisfy the condition (v) of Theorem 3.6.

(ii) In [5, Example (b)] an Orlicz function φ which does not satisfy the Δ_2-condition and is non-equivalent to a convex Orlicz function and yet satisfies the condition (v) of Theorem 3.6 is defined.

From Corollary 2.6 it follows that if φ satisfies the Δ_2-condition then the topological completion of $(L^\varphi(X), \tau_{L^\varphi(X)})$ equals $(L^{\overline{\varphi}}(X), \mathcal{T}_{\overline{\varphi}}(X))$. The next theorem tell us that if the conditions of Theorem 3.6 fail, then the topological completion of $(L^\varphi(X), \tau_{L^\varphi(X)})$ can not be treated as a function space in $L^0(X)$ (cf. [5, Proposition 5.6]). Let us recall that $L^0(X)$ can be provided with the complete metrizable topology $\mathcal{T}_0(X)$ of convergence in measure on sets of finite measure, i.e., $\mathcal{T}_0(X)$ is the topology of an F-norm $|f|_{L^0(X)} = \widetilde{f}_0$ for $f \in L^0(X)$, where $|\cdot|_0$ denotes the usual F-norm in L^0. Note that if $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$, then $\tau_{L^\varphi(X)} \supset \mathcal{T}_0(X)|_{L^\varphi(X)}$.

Theorem 3.7. *Assume that* $\liminf_{t\to\infty} \frac{\varphi(t)}{t} > 0$ *and the condition* (*v*) *of Theorem* 3.6 *does not hold. Then the natural continuous injection*

$$i\colon (L^\varphi(X), \tau_{L^\varphi(X)}) \hookrightarrow (L^0(X), \mathcal{T}_0(X))$$

can not be extended to a continuous injection from the topological completion $(\widehat{L^\varphi(X)}, \widehat{\tau_{L^\varphi(X)}})$ *of* $(L^\varphi(X), \tau_{L^\varphi(X)})$ *to* $L^0(X)$.

Proof. We will follow the lines of the proof of [5, Proposition 5.6], where the analogical result for an Orlicz sequence space ℓ^φ was obtained. Assume on the contrary that such a continuous extension is possible. It easily follows that if (f_n) is a $\tau_{L^\varphi(X)}$-Cauchy sequence in $L^\varphi(X)$ and $f_n \to 0$ for $\mathcal{T}_0(X)$, then $f_n \to 0$ for $\tau_{L^\varphi(X)}$. In view of Theorem 3.6 $L^\varphi(X) \cap E^{\overline{\varphi}}(X) \not\subset E^\varphi(X)$, so we can choose $f \in (L^\varphi(X) \cap E^{\overline{\varphi}}(X)) \setminus E^\varphi(X)$. Since supp $E^\varphi = \Omega$

there exists a sequence (Ω_n) in Σ such that $\Omega_n \uparrow \Omega$ and $\chi_{\Omega_n} \in E^\varphi$ (see [20, Theorem 86.2]). For $n = 1, 2, \ldots$ let us put

$$f^{(n)}(\omega) = \begin{cases} f(\omega) & \text{if} \quad \widetilde{f}(\omega) \\ \leqslant n \quad \text{and} \quad \omega \in \Omega_n, & \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f^{(n)} \in E^\varphi(X)$, and let $f_n = f - f^{(n)}$ for $n = 1, 2, \ldots$. Since $\widetilde{f_n} \downarrow 0$ in L^0 and $\mathcal{T}_0$ is a Lebesgue topology (see [1]) we conclude that $\widetilde{f_n} \to 0$ for $\mathcal{T}_0$, i.e., $f_n \to 0$ for $\mathcal{T}_0(X)$.

Moreover, since $f \in E^{\overline{\varphi}}(X)$, we see that $\widetilde{f_n} \downarrow 0$ in $E^{\overline{\varphi}}$, so $\widetilde{f_n} \to 0$ for $\mathcal{T}_{\overline{\varphi}|E^{\overline{\varphi}}}$, because $\mathcal{T}_{\overline{\varphi}|E^{\overline{\varphi}}}$ is a Lebesgue topology. Hence $f_n \to 0$ for $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$, so (f_n) is $\mathcal{T}_{\overline{\varphi}}(X)|_{L^\varphi(X)}$-Cauchy sequence, because $f_n \in L^\varphi(X)$ for $n = 1, 2, \ldots$. Clearly, $K_\varphi(f_n - f_m) = K_\varphi(f^{(n)} - f^{(m)}) = 0$ for $n, m = 1, 2, \ldots$, because $f^{(n)} \in E^\varphi(X)$. From Theorem 3.4 it follows that (f_n) is a $\tau_{L^\varphi(X)}$-Cauchy sequence. Thus according to the above remark $f_n \to 0$ for $\tau_{L^\varphi(X)}$.

On the other hand, since $f \notin E^\varphi(X) = \ker K_\varphi$ and $f^{(n)} \in E^\varphi(X)$ we get $0 < K_\varphi(f) = K_\varphi(f_n + f^{(n)}) \leqslant K_\varphi(f_n) + K_\varphi(f^{(n)}) = K_\varphi(f_n)$. It follows that $f_n \nrightarrow 0$ for $\tau_{L^\varphi(X)}$. ∎

REFERENCES

1. C. D. Aliprantis and O. Burkinshaw, Locally solid Riesz spaces, *Academic Press, New York,* 1978.

2. A. V. Bukhvalov, Vector-valued functions and tensor products. (Russian) *Siberian Math. J.*, **13**(1972), No. 6, 1229–1238.

3. S. Chen, H. Hudzik, On some convexities of Orlicz and Orlicz- Bochner spaces, *Comment. Math. Univ. Carolinae,* **29, 1**(1988), 13–29.

4. L. Drewnowski, On minimal topological linear spaces and strictly singular operators, *Comment. Math., T. Spec. in Honorem Ladislai Orlicz* **2**(1979), 80–106.

5. L. Drewnowski, M. Nawrocki, On the Mackey topology of Orlicz sequence spaces, *Archiv Math.* **39**(1982), 59–68.

6. L. Drewnowski, Compact operators on Musielak-Orlicz spaces, *Comment. Math.*, **27**(1988), 225–232.

7. K. Feledziak and M. Nowak, Locally solid topologies on vector-valued function spaces, *Collect. Math.* (to appear).

8. J. E. Jamison, A. Kamińska, and P.-K. Lin, Isometries of Orlicz spaces of vector valued functions, *Math. Z.* **193**(1986), 363–371.

9. N. J. Kalton, Orlicz sequence spaces without local convexity, *Math. Proc. Cambridge Phil. Soc.* **81**(1997), 253–277.

10. N. J. Kalton, Compact and strictly singular operators on Orlicz spaces, *Israel J. Math.* **26**(1977), 126–136.

11. L. Maligranda and W. Wnuk, Landau type theorem for Orlicz spaces, *Math. Z.* **208**(1991), 57–64.

12. W. Matuszewska and W. Orlicz, A note on the theory of s-normed spaces of φ-integrable functions, *Studia Math.* **21**(1961), 107–115.

13. W. Matuszewska and W. Orlicz, A note on modular spaces IX, *Bull. Acad. Polon. Sci.* **16**(1968), 801–808.

14. J. Musielak, Orlicz spaces and modular spaces, *Lecture Notes in Math.* 1034, *Springer-Verlag, Berlin, Heidelberg, New York, Tokyo,* 1983.

15. M. Nowak, Linear functionals on Orlicz sequence spaces without local convexity, *Inter. J. for Math. and Math. Sciences,* **15**(1992), No. 2, 241–254.

16. M. Nowak, Duality theory of non-locally convex Orlicz spaces, *Math. Japonica*, **38**(1993), No. 5, 813–832.

17. M. Nowak, Topological dual of non-locally convex Orlicz-Bochner spaces (to appear).

18. J. H. Shapiro, Mackey topologies reproducing kernels, and diagonal maps on the Hardy and Bergman Spaces, *Duke Math. J.* **43**(1976), 187–202.

19. P. Turpin, Convexités dans les espaces vectoriels topologiques généraux, *Dissertationes Math.* **131**(1976).

20. A. C. Zaanen, Riesz spaces II, *North-Holland Publ. Comp. Amsterdam, New York, Oxford,* 1983.

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