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ON THE RIEMANN PROBLEM WITH A MEASURABLE COEFFICIENT IN THE CLASS OF CAUCHY TYPE INTEGRALS WITH DENSITY FROM $L^{p(t)}$

1^0. Definitions and Notation

Let $p = p(t)$ be a positive function defined on a closed, simple, rectifiable curve Γ. We say that $p \in \mathcal{P}(\Gamma)$, if the conditions

1) $\exists\, C(p):\ \forall\, t_1, t_2 \in \Gamma,\ \big|p(t_1) - p(t_2)\big| < C(p)\big|\ln|t_1 - t_2|\big|^{-1}$, and

2) $\min p(t) = \underline{p} > 2$, are fulfilled.

By $L^{p(\cdot)}(\Gamma)$ we denote a set of those measurable on Γ functions for which

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_0^\ell \left|\frac{f(t(s))}{\lambda}\right|^{p(t(s))} ds < \infty\right\},$$

where $t = t(s)$, $0 \le s \le \ell$ is the equation of Γ with respect to the arc coordinate s.

Suppose

$$\widetilde{K}^{p(\cdot)}(\Gamma) = \bigg\{\Phi :\ \text{there is the polynomial} \quad Q_\Phi(z) : \Phi(z) = = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z} + Q_\Phi(z),\ \ \varphi \in L^{p(\cdot)}(\Gamma)\bigg\},$$

$$K^{p(\cdot)}(\Gamma) = \Big\{\Phi :\ \Phi \in \widetilde{K}^{p(\cdot)}(\Gamma), \quad Q_\Phi = 0\Big\}.$$

We say that the given on Γ measurable function G belongs to the class $A(p(t), \Gamma)$ if:

(i) $0 < m = \operatorname{ess\,inf}|G| \le \operatorname{ess\,sup}|G| = M < \infty$;

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(ii) for every point $\tau\in\Gamma$ there exists the arc $\Gamma_\tau\subset\Gamma$ such that almost all points $\{t,G(t)\}$, $t\in\Gamma_\tau$ lie inside of the angle with the vertex at the origin and opening

$$2\pi\Big[\sup_{\tau\in\Gamma_r}\big(\max(p(t),q(t))\Big]^{-1},\quad q(t)=p(t)\big(p(t)-1\big)^{-1}.$$

For $p(t)=\text{const}=p$, the class $A(p)$ coincides with the class A introduced by I. B. Simonenko [1] (see also [2]). The Riemann problem

$$\Phi^+(t)=G(t)\,\Phi^-(t)+g(t) \tag{1}$$

has also been considered by him in the class $K^p(\Gamma)$, when $G\in A$, $g\in L^p(\Gamma)$ and Γ is the Lyapunov's curve. The obtained by I.B. Simonenko results have been generalized in [3] (see also [4], Ch. II) for the case, where G belongs to a wider than A class of functions, and $\Gamma\in J^*$. Here $J^*=J_0\cap\Lambda$, where J_0 is the set of curves Γ with the equation $t=t(s)$, $0\le s\le\ell$ for which there exists the curve γ with the equation $t=\mu(s)$, $0\le s\le\ell$ such that

$$\operatorname*{ess\,sup}_{0\le\sigma\le\ell}\int\limits_0^\ell\left|\frac{t'(s)}{t(s)-t(\sigma)}-\frac{\mu(s)}{\mu(s)-\mu(\sigma)}\right|ds<\infty.$$

Λ is the set of Lavrentyev's curves, i.e., of curves Γ for which there exists the number M such that for every t_1, t_2 we have $s(t_1,t_2)\le M|t_1-t_2|$, where $s(t_1,t_2)$ is the least of two arcs lying on Γ and connecting the points t_1 and t_2.

Obviously, all smooth curves belong to J^*. The class J^* contains a set of those piecewise smooth and curves with bounded revolution which have no cusps (see [4], p. 23 and [5], p. 20). For the function $G\in A(p(t))$, just as for the constant p, we define the function $\arg G(t)$ and its index $\varkappa_G=\varkappa$.

2^0. Statement of the Problem

Let $\Gamma\in J^*$, $G(t)\in A(p(t))$, $p\in\mathcal{P}(\Gamma)$, $g(t)\in L^{p(\cdot)}(\Gamma)$; find the functions Φ from $K^{p(\cdot)}(\Gamma)$ for which the angular boundary values $\Phi^+(t)$ and $\Phi^-(t)$ satisfy almost for all $t\in\Gamma$ the condition (1).

3^0. The Basic Result

When solving the problem under consideration by the method of factorization, we reveal somewhat different picture of solvability than that for the constant p.

Assume

$$G_1(\tau) = (\tau - a)^{-\varkappa} G(\tau), \qquad a \in D^+,$$

$$X(z) = \begin{cases} \exp h(z), & z \in D^+, \\ (z-a)^{-\varkappa} h(z), & z \in D^-, \end{cases} \tag{2}$$

$$h(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\ln G_1(\tau)\, d\tau}{\tau - z}$$

and let

$$(T_g)(t) = \frac{X^+(t)}{2\pi i} \int\limits_{\Gamma} \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - t}, \quad t \in \Gamma. \tag{3}$$

Theorem. *If* $\Gamma \in J^*$, $p \in \mathcal{P}(\Gamma)$, $G \in A(p(t), \Gamma)$, $g \in L^{p(\cdot)}(\Gamma)$, $\operatorname{ind} G = 0$ *then the Riemann problem has a solution*

$$\Phi_g(z) = \frac{X(z)}{2\pi i} \int\limits_{\Gamma} \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - t} \tag{4}$$

belonging to the set

$$\bigcap_{g \in (0, \delta_0)} K^{p(t) - \delta}, \quad \delta_0 < p. \tag{5}$$

If, however, $G \in A(p(t), \Gamma)$, *then for the problem* (1) *to be solvable in the class* $K^{p(\cdot)}(\Gamma)$, *it is necessary that the condition*

$$(Tg)(t) \in L^{p(\cdot)}(\Gamma) \tag{6}$$

be fulfilled. When this condition is fulfilled, then:

(i) *for* $\varkappa = 0$, *the problem is uniquely solvable, and the solution is given by the equality* (4).

(ii) *for* $\varkappa > 0$, *the problem is solvable ambiguously, and all solution are given by the equality*

$$\Phi(z) = \Phi_g(z) + X(z)\, Q_{\varkappa - 1}(z), \tag{7}$$

where $Q_{\varkappa-1}$ *is an arbitrary polynomial of order* $\varkappa - 1$.

(iii) *for* $\varkappa < 0$, *for the problem to be solvable, it is necessary, in addition to the condition* (6), *and sufficient that the conditions*

$$\int\limits_{\Gamma} g(t) \big[X^+(t)\big]^{-1} t^k dt = 0, \quad k = 0, 1, \ldots, |\varkappa| - 1, \tag{8}$$

be fulfilled. If these conditions are fulfilled, then the solution is given by the equality (4).

4^0. About the Method Applied to the Investigation

In the course of our investigation we have applied the idea of reducing the problem (1) to a number of problems of similar type, but with the coefficient equal to the constant outside of a small arc lying on Γ. One of such methods, known for the constant p as "a local principle" ([1], [2]), is likewise valid for $p \in \mathcal{P}(\Gamma)$ (the proof can be obtained by the method mentioned in [6], by using the results from [7]–[9]). In order to apply the local principle, we have to find localizing classes for the case under consideration, and in case of a success, we would get a picture of solvability, leaving the problem of constructing a solution open. The suggested by us way makes it possible to construct solutions, if any. Towards this end, we have investigated thoroughly the operator T (the continuity in measure, closure in $L^{p(\cdot)}(\Gamma)$, compositions TS and ST, where S is the singular Cauchy operator, etc.)

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