**Proc. A. Razmadze Math. Inst. 162(2013), 136–140**

A. Kharazishvili

# ON SOME PATHOLOGICAL HOMOMORPHISMS OF UNCOUNTABLE COMMUTATIVE GROUPS

In what follows, $\mathbf{R}$ stands for the additive group of the real line and $\mathbf{T}$ stands for the additive group of the one-dimensional unit torus.

It is a well-known fact that if $(G, +)$ is an uncountable compact commutative group, then sometimes becomes possible to construct a homomorphism $\phi : G \to \mathbf{T}$ which possesses the following property: $\phi$ is nonmeasurable with respect to the completion $\mu$ of the Haar probability measure on $G$, but $\phi$ turns out to be measurable with respect to a certain translation invariant extension of $\mu$ (in this connection, see e.g. [6], [1], [3]–[5]).

Now, let $(G, +)$ be an uncountable commutative group not endowed with any topology. In general, we cannot assert that a homomorphism acting from $G$ into $\mathbf{T}$ (into $\mathbf{R}$) which has bad descriptive properties with respect to one group topology on $G$ is also bad with respect to another group topology on $G$.

The following simple example illustrates the said above.

**Example 1.** Consider the real line $\mathbf{R}$ and the Euclidean plane $\mathbf{R}^2$ as abstract commutative groups. As is well known, these two groups are isomorphic to each other. Let $\phi : \mathbf{R}^2 \to \mathbf{R}$ be such an isomorphism. Notice, by the way, that the existence of $\phi$ needs uncountable forms of the Axiom of Choice, because $\phi$ is a nonmeasurable function with respect to the ordinary two-dimensional Lebesgue measure $\lambda_2$ on $\mathbf{R}^2$ and, simultaneously, $\phi$ does not possess the Baire property (cf. [1], [5]). Briefly speaking, $\phi$ is bad from the point of view of the ordinary Euclidean topology on $\mathbf{R}^2$ and from the point of view of $\lambda_2$.

On the other hand, consider the bijection $\phi^{-1}$ and equip $\mathbf{R}^2$ with the topology $\phi^{-1}(\mathcal{T})$, where $\mathcal{T}$ is the standard Euclidean topology on $\mathbf{R}$. We thus obtain a locally compact topological group $(\mathbf{R}^2, \phi^{-1}(\mathcal{T}))$ such that $\phi$ turns out to be an isomorphism between $(\mathbf{R}^2, \phi^{-1}(\mathcal{T}))$ and $\mathbf{R}$, so $\phi$ has very good descriptive properties and, in particular, it is measurable with respect

2010 *Mathematics Subject Classification.* 28A20, 28D05, 22B99.

*Key words and phrases.* Commutative group, absolutely nonmeasurable homomorphism, universal measure zero set, Kulikov's theorem.

to the completion of the Haar measure on $(\mathbf{R}^2, \phi^{-1}(\mathcal{T}))$, which is the $\phi^{-1}$-image of the one-dimensional Lebesgue measure $\lambda_1$ on $\mathbf{R}$. Now, denote by $\psi : \mathbf{R} \to \mathbf{T}$ the canonical epimorphism given by

$$\psi(t) = (\cos(t), \sin(t)) \qquad (t \in \mathbf{R})$$

and take the composition $\chi = \psi \circ \phi$. Then it is easy to see that the said earlier is applicable to the homomorphism $\chi$, too.

This example inspires the question wether there are ultimately bad homomorphisms from an uncountable commutative group $(G, +)$ into $\mathbf{R}$ (or into $\mathbf{T}$). Here we discuss this question and describe all those commutative groups $(G, +)$ for which such homomorphisms exist.

Let us introduce some notation and several definitions.

For a given commutative group $(G, +)$, we shall denote by $\mathcal{M}(G)$ the class of all nonzero $\sigma$-finite translation quasi-invariant measures on $G$ (notice, by the way, that the domains of members of $\mathcal{M}(G)$ may be various translation invariant $\sigma$-algebras of subsets of $G$).

We shall say that a function $\phi$ acting from $G$ into $\mathbf{R}$ (into $\mathbf{T}$) is absolutely nonmeasurable with respect to the class $\mathcal{M}(G)$ if, for each measure $\mu \in \mathcal{M}(G)$, this $\phi$ is not measurable with respect to $\mu$.

Accordingly, we shall say that a set $X \subset G$ is absolutely nonmeasurable with respect to the class $\mathcal{M}(G)$ if the characteristic function (i.e., indicator) of $X$ is absolutely nonmeasurable with respect to $\mathcal{M}(G)$.

Let $Z \subset \mathbf{R}$ (respectively, $Z \subset \mathbf{T}$). We recall that $Z$ has universal measure zero if, for any $\sigma$-finite continuous Borel measure $\mu$ on $\mathbf{R}$ (respectively, on $\mathbf{T}$), the equality $\mu^*(Z) = 0$ holds, where $\mu^*$ denotes, as usual, the outer measure associated with $\mu$.

Some properties of universal measure zero sets are discussed in [5], [7], [9], and [10].

For our further purposes, the following auxiliary propositions are needed.

**Lemma 1.** *There exists, within* **ZFC** *set theory, an uncountable universal measure zero set $Z \subset \mathbf{R}$ (respectively, $Z \subset \mathbf{T}$) which simultaneously is a vector space over the field $\mathbf{Q}$ of all rational numbers.*

This lemma is well known (see, e.g., [9], [4], [5]).

**Lemma 2.** *Let $\phi$ be a homomorphism acting from a commutative group $(G, +)$ into $\mathbf{R}$ (into $\mathbf{T}$) such that the range of $\phi$ is an uncountable universal measure zero subset of $\mathbf{R}$ (of $\mathbf{T}$). Then $\phi$ is absolutely nonmeasurable with respect to the class $\mathcal{M}(G)$.*

**Lemma 3.** *Let $(G, +)$ be a commutative group, $G_0$ be its torsion subgroup and suppose that the quotient group $G/G_0$ is uncountable. Then there exists an uncountable subgroup $H$ of $G$ such that $G_0 \cap H = \{0\}$.*

**Lemma 4.** *Let $(G,+)$ be a commutative group and let $H$ be a subgroup of $G$. Then $H$ is not absolutely nonmeasurable with respect to the class $\mathcal{M}(G)$.*

**Lemma 5.** *Any commutative group $(\Gamma,+)$ can be represented in the form*

$$\Gamma = \cup\{\Gamma_n : n < \omega\},$$

*where the family $\{\Gamma_n : n < \omega\}$ is increasing by inclusion and all $\Gamma_n$ are direct sums of cyclic groups.*

Notice that Lemma 5 is due to Kulikov and is known as Kulikov's theorem on the algebraic structure of commutative groups. The proof of Lemma 5 can be found in [2] or [8].

We now are ready to formulate the main theorem.

**Theorem 1.** *Let $(G,+)$ be a commutative group and let $G_0$ be the torsion subgroup of $G$. The following two conditions are equivalent:*

(1) *the quotient group $G/G_0$ is uncountable;*

(2) *there exists a homomorphism from $G$ into $\mathbf{R}$ (into $\mathbf{T}$) which is absolutely nonmeasurable with respect to the class $\mathcal{M}(G)$.*

As a straightforward corollary of Theorem 1, we obtain

**Theorem 2.** *Let $(G,+)$ be a commutative group. The following two assertions are equivalent:*

(1) *there exists a homomorphism from $G$ into $\mathbf{R}$ absolutely nonmeasurable with respect to the class $\mathcal{M}(G)$;*

(2) *there exists a homomorphism from $G$ into $\mathbf{T}$ absolutely nonmeasurable with respect to the class $\mathcal{M}(G)$.*

Another immediate consequence of Theorem 1 is concerned with locally compact group topologies and is formulated as follows.

**Theorem 3.** *Let $(G,+)$ be a commutative group, $G_0$ be its torsion subgroup, and suppose that the quotient group $G/G_0$ is uncountable. Then there exists a homomorphism $\phi$ from $G$ into $\mathbf{R}$ (into $\mathbf{T}$) having the following property:*

*if $G$ is regarded as a thick subgroup of a $\sigma$-compact locally compact group $G'$ and $\mu$ is the measure induced on $G$ by the Haar measure $\mu'$ on $G'$, then $\phi$ turns out to be nonmeasurable with respect to any translation quasi-invariant extension of $\mu$.*

**Example 2.** Let $\omega$ denote the least infinite cardinal number and let $C = \{0,1\}^\omega$ be the Cantor space regarded as a commutative compact metrizable group with respect to the standard product topology and group operation modulo 2. By using the Continuum Hypothesis (Martin's axiom), it can be demonstrated that $C$ contains a Luzin subset (a generalized Luzin subset) $L$ which simultaneously is a subgroup of $C$. So, under these additional axioms, there exist universal measure zero subgroups of $C$ which are equinumerous with $C$. Let now $(G,+)$ be an arbitrary 2-divisible commutative group (e.g., $G=\mathbf{R}$ or $G=\mathbf{T}$). Then it is clear that any homomorphism $\phi : G \to C$ is trivial and, consequently, there exist no absolutely nonmeasurable homomorphisms acting from $G$ into $C$ (although condition (1) of Theorem 1 may be satisfied for $G$). At the same time, one can easily see that the identical embedding of $L$ into $C$ is a group monomorphism absolutely nonmeasurable with respect to the class of all nonzero $\sigma$-finite continuous measures on $L$ and, therefore, this embedding is also a group monomorphism absolutely nonmeasurable with respect to the class $\mathcal{M}(L)$. It should be noticed here that the class $\mathcal{M}(L)$ of measures is ample in the sense that, for every measure $\mu \in \mathcal{M}(L)$, there exists a measure $\mu' \in \mathcal{M}(L)$ which strictly extends $\mu$ (in this connection, see [3] or [5]).

In view of Theorem 1 and Example 2, the following problem arises.

**Problem.** Let $(G,+)$ be an uncountable commutative group and let $(H,+)$ be an uncountable commutative Polish topological group. Find necessary and sufficient conditions for the existence of an absolutely nonmeasurable homomorphism of $(G,+)$ into $(H,+)$.

Obviously, the analogous problem can be formulated for uncountable non-commutative groups.

## Acknowledgement

This research has been partially supported by Shota Rustaveli National Science Foundation, Grant N 31/25.

## References

1. W. W. Comfort, Topological groups. *Handbook of set-theoretic topology*, 1143–1263, *North-Holland, Amsterdam*, 1984.
2. L. Fuchs, Infinite abelian groups. Vol. I. Pure and Applied Mathematics, Vol. 36 *Academic Press, New York-London*, 1970.
3. A. B. Kharazishvili, Nonmeasurable sets and functions. North-Holland Mathematics Studies, 195. *Elsevier Science B.V., Amsterdam*, 2004.
4. A. B. Kharazishvili, On absolutely nonmeasurable additive functions. *Georgian Math. J.* **11** (2004), No. 2, 301–306.
5. A. B. Kharazishvili, Topics in Measure Theory and Real Analysis. *Atlantis Press and World Scientific. Amsterdam-Paris*, 2009.
6. K. Kodaira and S. Kakutani, A non-separable translation invariant extension of the Lebesgue measure space. *Ann. of Math. (2)* **52** (1950), 574–579.
7. K. Kuratowski, Topology. Vol. I. New edition, revised and augmented. Translated from the French by J. Jaworowski *Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw*, 1966.
8. A. G. Kurosh, The Theory of Groups. (Russian) *Nauka, Moscow*, 1967.

9. A. W. Miller, Special subsets of the real line. *Handbook of set-theoretic topology*, 201–233, *North-Holland, Amsterdam*, 1984.
10. J. C. Oxtoby, Measure and Category. *Springer-Verlag, New York*, 1971.

Author's address:

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177, Georgia
E-mail: kharaz2@yahoo.com