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J. Gvazava, R. Bitsadze, G. Baghaturia and M. Menteshashvili

# ON CHARACTERISTIC PROBLEMS WITH A NON-COMPACT SUPPORT OF DATA

There are various kinds of problems which are posed for hyperbolic equations. To such problems belong those when the data are given either on the open or on the closed support, the problems with the supports coming out from one point, the problems with the data on the intersecting supports, etc. If we take, for example, the equation of string oscillations and pose the values of the solution on the intersecting lines, then we can see that the data support is entirely inside of the area of definition of the solution. As was noted by Hadamard, such problems cannot be called boundary problems. Actually, they are called limit problems [1].

The segments of data support may be the segments of characteristics which are distant from each other. If we draw out the characteristics of both families coming out from the end points of these segments, using in some cases the principle of Asgeirsson, we can calculate values of the solution at the points of intersection of the characteristics. When the equation is nonlinear, the characteristic lines are unknown, and hence, when posing such problems we are given a free hand.

In this paper we consider some versions of the Goursat problem for which the data supports are not compact. We find particularly interesting and unexpected cases, where the support (or some part of it) remains outside of the area of definition of a solution. Considering these problems from such a point of view, we can compare them with the problems of control.

After the problem for the vibrating string is studied, we can consider the nonlinear equation:

$$u_{xx} + (1 + u_x + u_y) \cdot u_{xy} + (u_x + u_y) \cdot u_{yy} = 0, \qquad (1)$$

which has only linear characteristics. This class of equations is known to be quite wide [2]. It is easy to see that one family is given by the equation $x - y = c$. Taking into account that equation (1) along the second characteristic root is written in the form $d(u_x) + d(u_y) = 0$, the characteristics of this family are straight lines, and they are represented by the equation $dy =$

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$(u_x + u_y)dx$. Thus, the slope of characteristic lines depends on the sum of derivatives of unknown solutions [3].

It should be noted that despite the fact that the equation is hyperbolic, it degenerates parabolically when the relation $u_x + u_y = 1$ is fulfilled. Thus, the class of hyperbolic solutions is given by the condition $u_x + u_y \neq 1$, though we do not intend to consider only this class of solutions.

To pose the problem, we take the segment of fully defined characteristic line $I = \{(x, y) : y = x, x \in [0, a]\}$ and one of the conditions of the problem

$$u|_I = \varphi(x), \tag{2}$$

where $\varphi \in C^2(I)$ is the given function.

The characteristic lines of the second family, crossing the interval $I$, are completely determined by the condition $(u_x + u_y)|_I = \varphi'(x)$. Therefore, we can choose any segment of the following characteristics $\Gamma_{x_0} : y = x_0 + \varphi'(x_0)(x - x_0),\ \ x_0 \in [0, a]$ as a second segment of the data support. Let such a segment be $J = \{(x, y) : y = a + \varphi'(a)(x - a),\ \ x \in [b, c], b > a\}$ and let us consider the following

**Characteristic Problem 1.** Along with the domain of its definition, find a regular solution of (1) satisfying both the condition (2) and

$$u|_J = \psi(x), \tag{3}$$

where $\varphi \in C^2(I),\ \ \psi \in C^2(J)$, are the given functions.

The following theorem is true.

**Theorem 1.** *If the conditions*

$$\varphi'(a) < 1,\ -\infty < \varphi''(x) < 0,\ x \in [0, a]$$

*or*

$$\varphi'(a) > 1,\ 0 < \varphi''(x) < \infty,\ x \in [0, a],$$

*are fulfilled, then there exists the regular hyperbolic solution of the problem which is defined in the area $G$, and this area is bounded by the segments of straight lines $y = x + \Gamma_a(b) - b$, $y = x + \Gamma_a(c) - c$, $y = \varphi'(0)x$, $\Gamma_a(x)$.*

As we can see from the structure of the area $G$, one of the segments of the support of the problem (1)–(3), namely, the segment of the straight line $y = x$, remains outside of the area of definition of the solution. The support in the condition (3) is taken in such a way that it forms a part of the boundary of the area $G$.

An analogous question for the equation

$$u_y^4 u_{xx} - u_{yy} = cx^{-2} u u_y^4,\ \ c = const, \tag{4}$$

is considered which is of interest not only from the theoretical viewpoint, but as that having practical application [4]–[7].

The right-hand side of equation (4) along the $Oy-$ axis is unbounded. This property makes it possible to attribute this equation to the class of Euler–Darboux equations. Performing multiplication by a factor defining this unboundedness and making some assumptions, we can instead of (4) consider an equation with the degeneration not only of hyperbolicity but of order, too. However, there may exist solutions along which equation (4) remains hyperbolic. In other words, these are solutions for which the characteristic directions defined by the characteristic roots

$$\lambda_1 = u_y^{-2}, \quad \lambda_2 = -u_y^{-2},$$

do not coincide anywhere. Naturally, the class of hyperbolic solutions of equation (4) is defined by the condition $0 \neq |u_y(x,y)| < \infty$.

As is known, the characteristic roots $\lambda_1$ and $\lambda_2$ provide us with the differential relations of characteristic directions $u_y^2 dy - dx = 0$ and $u_y^2 dy + dx = 0$.

Suppose we are given two arcs $\gamma_1$ and $\gamma_2$ drawn from the common points $(a, f_1(a))$, $(a, f_2(a))$, $f_2(a) < f_1(a)$ and let them be given explicitly,

$$\gamma_1 : \ y = f_1(x), \quad a \leq x \leq b, \quad a > 0, \ \gamma_2 : \ y = f_2(x), \quad a \leq x \leq c.$$

Assume that the functions $f_1$ and $f_2$ are three times continuously differentiable and the arc $\gamma_1$ monotonically ascends, whereas the arc $\gamma_2$, vice versa, monotonically descends.

**Characteristic Problem 2.** Find a regular hyperbolic solution $u(x,y)$ of equation (4) and, simultaneously, a domain of its extension when the curves $\gamma_1$ and $\gamma_2$ are the arcs of the characteristics, and

$$u(a, f_1(a)) = \vartheta_1, \tag{5}$$

$$u_x(a, f_1(a)) = \delta_1, \tag{6}$$

$$u(a, f_2(a)) = \vartheta_2, \tag{7}$$

$$u_x(a, f_2(a)) = \delta_2. \tag{8}$$

By the conditions of the problem, we have $f_1'(x) = u_y^{-2}$, $f_2'(x) = -u_y^{-2}$. If

$$u_y|_{\gamma_1} = \frac{1}{\sqrt{f_1'(x)}}, \quad u_y|_{\gamma_2} = \frac{1}{\sqrt{-f_2'(x)}}$$

and parametrs $x_1 \in [a,b]$, $x_2 \in [a,c]$, then we can represent a solution of the problem (4)–(8) under consideration by the following three equalities:

$$x = F_1(x_1, x_2) \equiv \Big( 2\sqrt{-f_2'(x_2)}\, x_2^{1-\alpha} + 2\sqrt{-f_1'(x_1)}\, x_1^{1-\alpha} -$$

$$- \Big( \sqrt{f_1'(a)} + \sqrt{-f_2'(a)} + \delta_1 - \delta_2 \Big) a^{1-\alpha} + (\vartheta_1 - \vartheta_2)(1-\alpha) a^{-\alpha} \Big)^{\frac{1}{1-2\alpha}} \times$$

$$\times\Big(2\sqrt{-f_2'(x_2)}\,x_2^{\alpha}+2\sqrt{f_1'(x_1)}\,x_1^{\alpha}-$$

$$-\Big(\sqrt{f_1'(a)}+\sqrt{-f_2'(a)}+\delta_1-\delta_2\Big)a^{\alpha}+(\vartheta_1-\vartheta_2)\alpha a^{\alpha-1}\Big)^{\frac{1}{2\alpha-1}};\qquad(9)$$

$$y=G_1(x_1,x_2)\equiv f_1(x_1)+f_2(x_2)+$$

$$+\frac{1}{4(1-2\alpha)}\Big[4\sqrt{f_1'(x_1)}\,\sqrt{-f_2'(x_2)}\,(x_1^{1-\alpha}x_2^{\alpha}-x_1^{\alpha}x_2^{1-\alpha})+$$

$$+2\Big(\sqrt{-f_2'(x_2)}\,x_2^{1-\alpha}-\sqrt{f_1'(x_1)}\,x_1^{1-\alpha}\Big)\times$$

$$\times\Big(\Big(\sqrt{f_1'(a)}+\sqrt{-f_2'(a)}+\delta_1-\delta_2\Big)a^{\alpha}+\alpha(\vartheta_2-\vartheta_1)a^{\alpha-1}\Big)+$$

$$+2\Big(\sqrt{f_1'(x_1)}\,x_1^{\alpha}-\sqrt{-f_2'(x_2)}\,x_2^{\alpha}\Big)\times$$

$$\times\Big(\Big(\sqrt{f_1'(a)}+\sqrt{-f_2'(a)}+\delta_1-\delta_2\Big)a^{1-\alpha}+(1-\alpha)(\vartheta_2-\vartheta_1)a^{-\alpha}\Big)\Big];\qquad(10)$$

$$u=\frac{1}{1-2\alpha}\times$$

$$\times\Big[\Big(2\sqrt{-f_2'(x_2)}\,x_2^{\alpha}-\Big(\sqrt{-f_2'(a)}-\delta_2\Big)a^{\alpha}-\alpha\vartheta_2a^{\alpha-1}\Big)F_1^{1-\alpha}(x_1,x_2)+$$

$$+\Big(2\sqrt{-f_2'(x_2)}\,x_2^{1-\alpha}+$$

$$+\Big(\sqrt{-f_2'(a)}-\delta_2\Big)a^{1-\alpha}-(1-\alpha)\vartheta_2a^{-\alpha}\Big)F_1^{\alpha}(x_1,x_2)\Big].\qquad(11)$$

The above equations define simultaneously both the definition domain $D_1$ and the unknown solution.

If

$$u_y|_{\gamma_1}=-\frac{1}{\sqrt{f_1'(x)}},\quad u_y|_{\gamma_2}=-\frac{1}{\sqrt{f_2'(x)}}$$

then we can see that the problem (4)–(8) has the solution represented by the following formulas:

$$x=F_2(x_1,x_2)\equiv\Big(2\sqrt{-f_2'(x_2)}\,x_2^{1-\alpha}+2\sqrt{f_1'(x_1)}\,x_1^{1-\alpha}-$$

$$-\Big(\sqrt{f_1'(a)}+\sqrt{-f_2'(a)}+\delta_2-\delta_1\Big)a^{1-\alpha}+(\vartheta_2-\vartheta_1)(1-\alpha)a^{-\alpha}\Big)^{\frac{1}{1-2\alpha}}\times$$

$$\times\Big(2\sqrt{-f_2'(x_2)}\,x_2^{\alpha}+2\sqrt{f_1'(x_1)}\,x_1^{\alpha}-$$

$$-\Big(\sqrt{f_1'(a)}+\sqrt{-f_2'(a)}+\delta_2-\delta_1\Big)a^{\alpha}+(\vartheta_2-\vartheta_1)\alpha a^{\alpha-1}\Big)^{\frac{1}{2\alpha-1}};\qquad(12)$$

$$y = G_2(x_1, x_2) \equiv f_1(x_1) + f_2(x_2) +$$
$$+\frac{1}{4(1-2\alpha)}\Big[4\sqrt{f_1'(x_1)}\sqrt{-f_2'(x_2)}\,(x_1^{1-\alpha}x_2^{\alpha} - x_1^{\alpha}x_2^{1-\alpha}) -$$
$$-2\Big(\sqrt{-f_2'(x_2)}\,x_2^{1-\alpha} - \sqrt{f_1'(x_1)}\,x_1^{1-\alpha}\Big)\times$$
$$\times\Big(\Big(\delta_1 - \delta_2 - \sqrt{f_1'(a)} - \sqrt{f_2'(a)}\Big)a^{\alpha} + \alpha(\vartheta_2 - \vartheta_1)a^{\alpha-1}\Big) -$$
$$-2\Big(\sqrt{f_1'(x_1)}\,x_1^{\alpha} - \sqrt{-f_2'(x_2)}\,x_2^{\alpha}\Big)\times$$
$$\times\Big(\Big(-\sqrt{f_1'(a)} - \sqrt{-f_2'(a)} + \delta_1 - \delta_2\Big)a^{\alpha} + \alpha(\vartheta_2 - \vartheta_1)a^{\alpha-1}\Big)\Big]; \qquad (13)$$
$$u = \frac{1}{2\alpha - 1}\times$$
$$\times\Big[\Big(2\sqrt{-f_2'(x_2)}\,x_2^{\alpha} - \Big(\delta_2 - \sqrt{-f_2'(a)}\Big)a^{\alpha} + \alpha\vartheta_2 a^{\alpha-1}\Big)F_2^{1-\alpha}(x_1, x_2) +$$
$$+\Big(2\sqrt{-f_2'(x_2)}\,x_2^{1-\alpha} +$$
$$+\Big(\sqrt{-f_2'(a)} - \delta_2\Big)a^{1-\alpha} + (1-\alpha)\vartheta_2 a^{-\alpha}\Big)F_2^{\alpha}(x_1, x_2)\Big]; \qquad (14)$$

equalities (12),(13) define the definition domain $D_2$.

Suppose

$$[F_1(x_1^0, x_2) - F_1(x_1', x_2)]^2 + [G_1(x_1^0, x_2) - G_1(x_1', x_2)]^2 \neq 0, \qquad (15)$$
$$[F_1(x_1, x_2^0) - F_1(x_1, x_2')]^2 + [G_1(x_1, x_2^0) - G_1(x_1, x_2')]^2 \neq 0, \qquad (16)$$
$$[F_2(x_1^0, x_2) - F_2(x_1', x_2)]^2 + [G_2(x_1^0, x_2) - G_2(x_1', x_2)]^2 \neq 0, \qquad (17)$$
$$[F_2(x_1, x_2^0) - F_2(x_1, x_2')]^2 + [G_2(x_1, x_2^0) - G_2(x_1, x_2')]^2 \neq 0, \qquad (18)$$

for any fixed values $x_1^0 \neq x_1',\ x_1^0, x_1' \in [a, b]$ and $x_2^0 \neq x_2',,\ x_2^0, x_2' \in [a, c]$ and for the parametrs $x_1 \in [a, b]$, $x_2 \in [a, c]$.

The following theorem is true.

**Theorem 2.** *Under the conditions* (15)–(18)*, for any real branch of the multi-valued functions* (9)*,* (12) *there are regular hyperbolic solutions of the problem* (4)–(8) *represented explicitly by the formulas* (9)–(11) *in the domain* $D_1$ *and by* (12)–(14) *in the domain* $D_2$*, where the domain* $D_1$ *is bounded by the arcs of characteristic curves*

$$\Gamma_1:\ x = F_1(a, x_2),\ \ y = G_1(a, x_2);\ \ \Gamma_2:\quad x = F_1(x_1, c),\ \ y = G_1(x_1, c);$$
$$\Gamma_3:\ x = F_1(b, x_2),\ \ y = G_1(b, x_2);\ \ \Gamma_4:\quad x = F_1(x_1, d),\ \ y = G_1(x_1, d);$$

*and the domain $D_2$ is bounded by the arcs of characteristic curves*

$$\Gamma_5: \ x = F_2(a, x_2), \ \ y = G_2(a, x_2); \ \Gamma_6: \quad x = F_2(x_1, c), \ \ y = G_2(x_1, c);$$
$$\Gamma_7: \ x = F_2(b, x_2), \ \ y = G_2(b, x_2); \ \Gamma_8: \quad x = F_2(x_1, d), \ \ y = G_2(x_1, d);$$

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Authors' addresses:

J. Gvazava

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177, Georgia

R. Bitsadze

Georgian Technical University
77, M. Kostava St., Tbilisi 0175, Georgia

G. Baghaturia and M. Menteshashvili

N. I. Muskhelishvili Computational Mathematics Institute of Georgian

Technical University, 8 Akuri St., Tbilisi 0160, Georgia