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REPRESENTING SUMMABLE FUNCTIONS OF TWO VARIABLES BY DOUBLE EXPONENTIAL FOURIER SERIES

Every function f of two variables that is 2π-periodic in each variable and summable in $I = [0, 2\pi]^2$, or shortly $f \in L(I)$, has its double exponential Fourier series

$$f \sim c_{00} + \sum_{|m|\geq 1} c_{m0}e^{imx} + \sum_{|n|\geq 1} c_{0n}e^{inx} + \sum_{|m|\geq 1, |n|\geq 1} c_{mn}e^{i(mx+ny)}. \qquad (1)$$

It is well-known ([1]) that there is a continuous function F on I, for which the series (1) is everywhere divergent; that is, for each point $(x, y) \in I$, the equality

$$F(x, y) = \lim_{M,N\to+\infty} \sum_{|m|\leq M, |n|\leq N} c_{mn}e^{i(mx+ny)} \qquad (2)$$

fails to hold. This is so even for those continuous function on I, for which $c_{m,n=0}$, when $m < 0$ or $n < 0$ and $\frac{1}{\lambda} \leq \frac{M}{N} \leq \lambda$ with $\lambda > 1$ (see [2]).

For this reason, it is desirable to find "means" associated with the series (1), such that they will represent the function f. And we do this by showing that the equality

$$\int_o^x \int_0^y f(t, \tau)dtd\tau = c_{00}xy + iy \sum_{|m|\geq 1} \frac{c_{m0}}{m}(1-e^{imx}) + ix \sum_{|n|\geq 1} \frac{c_{0n}}{n}(1-e^{inx}) - \\ - \sum_{|m|\geq 1, |n|\geq 1} \frac{c_{mn}}{mn}(1-e^{imx})(1-e^{inx}) \qquad (3)$$

holds uniformly in I, even when the series (1) is everywhere divergent.

According to a result due to Lebesgue (see, for example, [3]), if

$$\lambda \sim c_0 + \sum_{|k|\geq 1} c_k e^{ikx} \qquad (4)$$

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is a series corresponding to a one-variable 2π-periodic function $\lambda(t)$ that is summable in the interval $[0, 2\pi]$, then

$$\int_0^x \lambda(t)dt = c_0 x + i \sum_{|k|\geq 1} \frac{c_k}{k}(1 - e^{ikx}). \tag{5}$$

Note that (3) extends the last equality to the case of the double exponential Fourier series (1). Note also that the series (4) may be divergent everywhere. The existence of such a series was shown by A. Kolmogorov (e.g. [3]).

From (3) we obtain:

$$\int_o^x \int_0^y f(t,\tau)dtd\tau = A(x,y) + B(x,y), \tag{6}$$

where

$$A(x,y) = c_{00}xy - iy \sum_{|m|\geq 1} \frac{1}{m} c_{m0} e^{imx} - ix \sum_{|n|\geq 1} \frac{1}{n} c_{0n} e^{iny} - \\ - \sum_{|m|\geq 1, |n|\geq 1} \frac{1}{mn} c_{mn} e^{i(mx+ny)} \tag{7}$$

and

$$B(x,y) = iy \sum_{|m|\geq 1} \frac{1}{m} c_{m0} + ix \sum_{|n|\geq 1} \frac{1}{n} c_{0n} - \sum_{|m|\geq 1, |n|\geq 1} \frac{1}{mn} c_{mn} + \\ + \sum_{|m|\geq 1, |n|\geq 1} \frac{1}{mn} c_{mn} e^{imx} + \sum_{|m|\geq 1, |n|\geq 1} \frac{1}{mn} c_{mn} e^{iny}. \tag{8}$$

The right hand side of the equality (7) is the series obtained by integrating formally the series (1) with respect to (x, y) and is related to the following two means (in the sense of Lebesgue) associated to the series (1):

$$L_{h,k}(f; x, y) = c_{00} + \sum_{|m|\geq 1} c_{m0} e^{imx} \frac{\sin mh}{mh} + \sum_{|n|\geq 1} c_{0n} e^{iny} \frac{\sin nk}{nk} + \\ + \sum_{|m|\geq 1, |n|\geq 1} c_{mn} e^{i(mx+ny)} \cdot \frac{\sin mh}{mh} \cdot \frac{\sin nk}{nk} \tag{9}$$

and

$$L^*_{h,k}(f;x,y) = c_{00} + \sum_{|m|\geq 1} c_{m0} e^{imx} e^{imh} e^{imk} \frac{\sin mh}{mh} + \sum_{|n|\geq 1} c_{0n} e^{iny} \frac{\sin nk}{nk} + $$
$$+ \sum_{|m|\geq 1, |n|\geq 1} c_{mn} e^{i(mx+ny)} e^{i(mh+nk)} \cdot \frac{\sin mh}{mh} \cdot \frac{\sin nk}{nk}. \quad (10)$$

Proofs:

(1) If $f \in L(I)$ is continuous at a point $(x,y) \in I$, then we have

$$f(x,y) = \lim_{(h,k)\to(0,0)} L_{h,k}(f;x,y) \quad (11)$$

also

$$f(x,y) = \lim_{(h,k)\to(0,0)} L^*_{h,k}(f;x,y). \quad (12)$$

(2) If $f \in \ln^+ |f| \in L(I)$, then the equalities (11) and (12) are satisfied for almost all points $(x,y) \in I$. In particular, this is the case for $f \in L^p(I)$ with $p > 1$.

(3) When $\frac{1}{\lambda} \leq \frac{|h|}{|k|} \leq \lambda$ and $\lambda > 1$, then for every function $f \in L(I)$, the equalities (11) and (12) are satisfied for almost all points $(x,y) \in I$.

(4) For every function $f \in L(I)$, the following equalities hold

$$f(x,y) = \lim_{k\to 0} \lim_{h\to 0} L_{h,k}(f;x,y) \quad (13)$$

also

$$f(x,y) = \lim_{h\to 0} \lim_{k\to 0} L^*_{h,k}(f;x,y) \quad (14)$$

for almost all points $(x,y) \in I$. Analogous equalities hold for $L^*_{h,k}(f;x,y)$.

References

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