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MIXED BOUNDARY VALUE PROBLEMS OF DIFFRACTION BY A HALF-PLANE WITH A SCREEN/CRACK PERPENDICULAR TO THE BOUNDARY

1. Introduction and Formulation of the Problems

We will analyse certain classes of wave diffraction problems which will be formulated as mixed boundary value problems for the Helmholtz equation in a domain with a crack. This will be done in a Sobolev space setting which is chosen taking into account both physical and mathematical arguments.

In order to define the classes of problems in a rigorous way, we start by establishing the general notation which will allow the mathematical formulation of the problem. As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L_2(\mathbb{R}^n)}$ is finite. As the notation indicates, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$.

For a given Lipschitz domain $\mathcal{D}$, on $\mathbb{R}^n$, we denote by $\widetilde{H}^s(\mathcal{D})$ the closed subspace of $H^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to $H^s(\mathbb{R}^n)$. The space $\widetilde{H}^s(\mathcal{D})$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we introduce the norm of the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\mathbb{R}^n\backslash\overline{\mathcal{D}})$. Throughout the paper we will use the notation $\mathbb{R}^n_\pm := \{x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n : \pm x_n > 0\}$. Note that the spaces $H^0(\mathbb{R}^n_+)$ and $\widetilde{H}^0(\mathbb{R}^n_+)$ can be identified, and we will denote them by $L_2(\mathbb{R}^n_+)$.

Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 \in \mathbb{R}\}$, $\Gamma_1 := \{(x_1, 0) : x_1 \in \mathbb{R}\}$, and $\Gamma_2 := \{(0, x_2) : x_2 \in \mathbb{R}\}$. Let further $\mathcal{C} := \{(x_1, 0) : 0 < x_1 < a\} \subset \Gamma_1$ for a certain positive number a and $\Omega_{\mathcal{C}} := \Omega\backslash\overline{\mathcal{C}}$. Clearly, $\partial\Omega = \Gamma_2$ and

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$\partial\Omega_{\mathcal{C}} = \Gamma_2 \cup \mathcal{C}$. For our purposes below we introduce further notations: $\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and $\Omega_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\}$; then, $\partial\Omega_j = \mathcal{S}_j \cup \mathcal{S}$, for $j = 1, 2$, where $\mathcal{S} := \{(x_1, 0) : x_1 \geq 0\} \subset \Gamma_1$, $\mathcal{S}_1 := \{(0, x_2) : x_2 \geq 0\} \subset \Gamma_2$, and $\mathcal{S}_2 := \{(0, x_2) : x_2 \leq 0\} \subset \Gamma_2$. Finally, we introduce the following unit normal vectors $n_1 = \overrightarrow{(0, -1)}$ on Γ_1 and $n_2 = \overrightarrow{(-1, 0)}$ on Γ_2.

Let $\varepsilon \in [0, \frac{1}{2})$. We are interested in studying the problem of existence and uniqueness of an element $u \in H^{1+\varepsilon}(\Omega_{\mathcal{C}})$, such that

$$\left(\Delta + k^2\right) u = 0 \quad \text{in} \quad \Omega_{\mathcal{C}}, \tag{1}$$

and u satisfies one of the following two boundary conditions:

$$[u]^{\pm}_{\mathcal{C}} = g_0^{\pm} \quad \text{on } \mathcal{C}, \quad [u]^{+}_{\mathcal{S}_1} = h_1 \quad \text{on } \mathcal{S}_1, \quad [\partial_{n_2} u]^{+}_{\mathcal{S}_2} = f_2 \quad \text{on } \mathcal{S}_2, \tag{2}$$

and

$$[\partial_{n_1} u]^{\pm}_{\mathcal{C}} = g_1^{\pm} \quad \text{on } \mathcal{C}, \quad [\partial_{n_2} u]^{+}_{\mathcal{S}_1} = f_1 \quad \text{on } \mathcal{S}_1, \quad [u]^{+}_{\mathcal{S}_2} = h_2 \quad \text{on } \mathcal{S}_2, \tag{3}$$

Here, the wave number $k \in \mathbb{C} \setminus \mathbb{R}$ is given. The elements $[u]^{+}_{\mathcal{S}_j}$ and $[\partial_{n_2} u]^{+}_{\mathcal{S}_j}$ denote the Dirichlet and the Neumann traces on $\mathcal{S}_j$, respectively, while by $[u]^{\pm}_{\mathcal{C}}$ we denote the Dirichlet traces on $\mathcal{C}$ from both sides of the screen and by $[\partial_{n_1} u]^{\pm}_{\mathcal{C}}$ we denote the Neumann traces on $\mathcal{C}$ from both sides of the crack.

Throughout the paper on the given data we assume that $h_j \in H^{1/2+\varepsilon}(\mathcal{S}_j)$, $f_j \in H^{-1/2+\varepsilon}(\mathcal{S}_j)$, for $j = 1, 2$ and $g_i^{\pm} \in H^{1/2-i+\varepsilon}(\mathcal{C})$, for $i = 0, 1$. Furthermore, we suppose that they satisfy the following compatibility conditions:

$$\chi_a(g_0^+ - g_0^-) \in r_{\mathcal{C}} \widetilde{H}^{1/2+\varepsilon}(\mathcal{C}), \tag{4}$$

$$\chi_a(g_1^+ - g_1^-) \in r_{\mathcal{C}} \widetilde{H}^{-1/2+\varepsilon}(\mathcal{C}). \tag{5}$$

$$\chi_0 \left(g_0^+ - r_{\mathcal{C}} h_1 \circ e^{i\frac{\pi}{2}}\right) \in r_{\mathcal{C}} \widetilde{H}^{1/2+\varepsilon}(\mathcal{C}), \tag{6}$$

$$\chi_0 \left(g_1^- - r_{\mathcal{C}} f_1 \circ e^{-i\frac{\pi}{2}}\right) \in r_{\mathcal{C}} \widetilde{H}^{-1/2+\varepsilon}(\mathcal{C}). \tag{7}$$

Here, $r_{\mathcal{C}}$ denotes the restriction operator to $\mathcal{C}$ and $\chi_a(x) := \chi_0(a - x)$, where $\chi_0 \in C^\infty([0, a])$, such that $\chi_0(x) \equiv 1$ for $x \in [0, a/3]$ and $\chi_0(x) \equiv 0$ for $x \in [2a/3, a]$. From now on we will refer to:

- Problem $\mathcal{P}_{D-M}$ as the problem characterized by (1), (2), (4), and (6);
- Problem $\mathcal{P}_{N-M}$ as the one characterized by (1), (3), (5), and (7).

Note that the just stated compatibility conditions are necessary conditions to the well-posedness of the corresponding problems. Note also that, the compatibility conditions (5) and (7) included in Problem $\mathcal{P}_{N-M}$ are additional restrictions only for $\varepsilon = 0$.

2. Uniqueness, Existence and Regularity Results

We start this section by mentioning the uniqueness result for the problems in consideration.

Theorem 2.1. *The problems $\mathcal{P}_{D-M}$ and $\mathcal{P}_{N-M}$ have at most one solution each.*

The proof is standard and uses the Green formula, cf. [2].

Now, without lost of generality, we assume that $\Im\mathrm{m}\, k > 0$; the complementary case $\Im\mathrm{m}\, k < 0$ runs with obvious changes. Let us denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by $\mathcal{K}(x) := -\frac{i}{4}H_0^{(1)}(k|x|)$, where $H_0^{(1)}$ is the Hankel function of the first kind of order zero (cf. [3, §3.4]). Denote by V_j and W_j the single and the double layer potentials on Γ_j, respectively:

$$\begin{aligned} V_j(\psi)(x) &= \int\limits_{\Gamma_j} \mathcal{K}(x-y)\psi(y)d_y\Gamma_j\ , \quad x \notin \Gamma_j\ , \\ W_j(\varphi)(x) &= \int\limits_{\Gamma_j} [\partial_{n_j(y)}\mathcal{K}(x-y)]\varphi(y)d_y\Gamma_j\ , \quad x \notin \Gamma_j\ , \end{aligned}$$

where $j = 1, 2$ and ψ, φ are density functions. Furthermore, we introduce the even and odd extension operators defined by

$$\ell^e\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ \varphi(-y), & y \in \mathbb{R}_\mp \end{cases} \quad \text{and} \quad \ell^o\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ -\varphi(-y), & y \in \mathbb{R}_\mp \end{cases},$$

respectively.

The boundary value problem $\mathcal{P}_{D-M}$ can be equivalently rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j = 1, 2$, such that

$$\left(\Delta + k^2\right) u_j = 0 \quad \text{in} \quad \Omega_j, \tag{8}$$

$$[u_1]^+_{\mathcal{S}_1} = h_1 \quad \text{on } \mathcal{S}_1, \quad [\partial_{n_2}u_2]^+_{\mathcal{S}_2} = f_2 \quad \text{on} \quad \mathcal{S}_2, \tag{9}$$

$$[u_1]^+_{\mathcal{C}} = g_0^+, \quad [u_2]^-_{\mathcal{C}} = g_0^- \quad \text{on} \quad \mathcal{C}, \tag{10}$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \quad [\partial_{n_1}u_1]^+_{\mathcal{C}^c} - [\partial_{n_1}u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on} \quad \mathcal{C}^c, \tag{11}$$

where $\mathcal{C}^c = \mathcal{S}\backslash\overline{\mathcal{C}}$.

Let us consider the following functions (cf. [1]):

$$u_1 = 2W_1\left(\ell^o(\ell_+g_0^+ - [2W_2(\ell^e h_1)]^+_{\mathcal{S}}) + \ell^o(r_{\mathcal{S}}\varphi)\right) + 2W_2(\ell^e h_1) \quad \text{in } \Omega_1 \tag{12}$$

and

$$u_2 = -2W_1\left(\ell^e(\ell_+g_0^- + r_{\mathcal{S}}\varphi)\right) + 2V_2(\ell^o f_2) \quad \text{in } \Omega_2, \tag{13}$$

where φ is an arbitrary element of the space $\widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\ell_+g_0^+ \in H^{\frac{1}{2}+\varepsilon}(\mathcal{S})$ is any fixed extension of $g_0^+ \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C})$, while $\ell_+g_0^- \in H^{\frac{1}{2}+\varepsilon}(\mathcal{S})$ denotes the extension of $g_0^- \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C})$ which satisfies the condition $r_{\mathcal{C}^c}(\ell_+ g_0^+ - \ell_+ g_0^-) = 0$. Note that such extension exists due to the compatibility condition (4). Note also that, the compatibility conditions (6) ensure us that $\ell_+ g_0^+ - [2W_2(\ell^e h_1)]_{\mathcal{S}}^+$ is an element of $r_{\mathcal{S}} \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{S})$ and, therefore, we may apply the extension operator ℓ^o.

It is easy to verify that u_j belong to the spaces $H^{1+\varepsilon}(\Omega_j)$ and satisfy equations (8)-(10). Moreover, on $\mathcal{C}^c$, we have

$$[u_1]_{\mathcal{C}^c}^+ - [u_2]_{\mathcal{C}^c}^- = 0.$$

Therefore, it remains to satisfy the condition

$$[\partial_{n_1} u_1]_{\mathcal{C}^c}^+ - [\partial_{n_1} u_2]_{\mathcal{C}^c}^- = 0,$$

which together with (12) and (13) leads us to the following equation

$$r_{\mathcal{C}^c} \mathcal{L} \varphi = F^D, \tag{14}$$

where $F^D := \frac{1}{2} r_{\mathcal{C}^c} \big([\partial_{n_1} V_2(\ell^o f_2)]_{\mathcal{S}}^- - \mathcal{L}(\ell^o(\ell_+ g_0^+ - [2W_2(\ell^e h_1)]_{\mathcal{S}}^+) + \ell^e \ell_+ g_0^-) \big)$.

Thus, we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c} \mathcal{L} : \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c),$$

which, actually, is an invertible operator, cf. [4].

The boundary value problem $\mathcal{P}_{N-M}$ can be equivalently rephrased in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, $j = 1, 2$, such that

$$\left(\Delta + k^2\right) u_j = 0 \quad \text{in} \quad \Omega_j, \tag{15}$$

$$[u_1]_{\mathcal{S}_1}^+ = h_1 \quad \text{on } \mathcal{S}_1, \quad [\partial_{n_2} u_2]_{\mathcal{S}_2}^+ = f_2 \quad \text{on} \quad \mathcal{S}_2, \tag{16}$$

$$[\partial_{n_1} u_1]_{\mathcal{C}}^+ = g_1^+, \quad [\partial_{n_1} u_2]_{\mathcal{C}}^- = g_1^- \quad \text{on} \quad \mathcal{C}, \tag{17}$$

and

$$[u_1]_{\mathcal{C}^c}^+ - [u_2]_{\mathcal{C}^c}^- = 0, \quad [\partial_{n_1} u_1]_{\mathcal{C}^c}^+ - [\partial_{n_1} u_2]_{\mathcal{C}^c}^- = 0 \quad \text{on} \quad \mathcal{C}^c,$$

where $\mathcal{C}^c = \mathcal{S} \backslash \overline{\mathcal{C}}$.

We shall consider the following functions (cf. [1]):

$$u_1 = 2W_2(\ell^e h_1) - 2V_1(\ell^o(\ell_+ g_1^+ + r_{\mathcal{S}} \psi)) \quad \text{in} \quad \Omega_1 \tag{18}$$

and

$$u_2 = -2V_2(\ell^o f_2) + 2V_1(\ell^e(\ell_+ g_1^- + 2[\partial_{n_1} V_2(\ell^o f_2)]_{\mathcal{S}}^-) + \ell^e(r_{\mathcal{S}} \psi)) \text{ in } \Omega_2, \tag{19}$$

where ψ is an arbitrary element of the space $\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\ell_+ g_1^+ \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ is any fixed extension of $g_1^+ \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C})$, while $\ell_+ g_1^- \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ denotes the extension of $g_1^- \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C})$ which satisfies the condition $r_{\mathcal{C}^c}(\ell_+ g_1^+ - \ell_+ g_1^-) = 0$. Note that such extension exists due to the compatibility condition (5).

It is straightforward to verify that u_j belong to the spaces $H^{1+\varepsilon}(\Omega_j)$ and satisfy equations (15)–(17). Moreover, on $\mathcal{C}^c$, we have

$$[\partial_{n_1} u_1]^+_{\mathcal{C}^c} - [\partial_{n_1} u_2]^-_{\mathcal{C}^c} = 0.$$

Thus, it remains to satisfy the condition

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0,$$

which together with (18) and (19) leads us to the following equation

$$r_{\mathcal{C}^c}\mathcal{H}\psi = F^N, \tag{20}$$

where $F^N = \frac{1}{2} r_{\mathcal{C}^c}\big([W_2(\ell^e h_1)]^+_{\mathcal{S}} - \mathcal{H}(\ell^o(\ell_+ g_1^+) + \ell^e(\ell_+ g_1^- + 2[\partial_{n_1} V_2(\ell^o f_2)]))\big)$. Consequently, the analysis of this last problem is equivalently reduced to the investigation of the invertibility of the operator

$$r_{\mathcal{C}^c}\mathcal{H} : \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c),$$

which in fact is an invertible operator for the space smoothness orders in consideration; cf. [4].

Due to a direct combination of the results above, we now obtain the main conclusion of the present work for the problems in analysis.

Theorem 2.2. *Let* $0 \le \varepsilon < \frac{1}{2}$.

(i) *The Problem* $\mathcal{P}_{D-M}$ *has a unique solution which is representable as a pair* (u_1, u_2) *defined by the formulas* (12), (13), *where* φ *is the unique solution of the equation* (14).

(ii) *The Problem* $\mathcal{P}_{N-M}$ *has a unique solution which is representable as a pair* (u_1, u_2) *defined by the formulas* (18), (19), *where* ψ *is the unique solution of the equation* (20).

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