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ON SOME SUFFICIENT STABILITY CONDITIONS OF NONISOTHERMAL FLOW BETWEEN POROUS CYLINDERS

Let a viscous heat-conducting fluid fill the cavity between two rotating permeable cylinders heated up to different temperatures. We denote the radii, angular velocities and temperatures of the internal and the outer cylinders by R_1, Ω_1, Θ_1 and R_2, Ω_2, Θ_2, respectively. It is assumed that external mass forces are absent, the velocity of the flow through the cylinder cavity cross-section is equal to zero, and the fluid inflow through the walls of one cylinder is equal to the fluid outflow through the walls of the other.

Let the scales of length, velocity, time, temperature, pressure and density be denoted, respectively, by R_1, $\Omega_1 R_1$, $1/\Omega_1$, Θ_1, $\nu\rho'\Omega_1$, the density scale be as the fluid density at the temperature Θ_1. Using the dimensionless system of Navier-Stokes, heat transfer, continuity equations and an equation of state, under the above assumptions we obtain - in terms of cylindrical coordinates (r, φ, z) with the z-axis coinciding with that of the cylinders - the following exact solution with the velocity vector $\boldsymbol{V}_0 = \{v_{0r}(r), v_{0\varphi}(r), 0\}$, temperature T_0, pressure Π_0 [1]:

$$
\begin{gathered}
v_{0r} = \frac{\varkappa}{\operatorname{Re} r}, \quad v_{0\varphi} = \begin{cases} ar^{\varkappa+1} + \dfrac{b}{r}, & \varkappa \neq 2, \\ \dfrac{a_1 \ln r + 1}{r}, & \varkappa = -2, \end{cases} \qquad T_0 = c_1 + c_2 r^{\varkappa P_r}, \\
\Pi_0 = \int\limits_1^r \left(\frac{v_{0\varphi}^2}{r} - \frac{\varkappa^2}{\operatorname{Re}^2 r^3} \right) \Big[1 - \frac{Ra}{\Pr} (r^{\varkappa \Pr} - 1) \Big] dr + \text{const},
\end{gathered} \tag{1}
$$

where

$$
a = \frac{\Omega R^2 - 1}{R^{\varkappa+2} - 1}, \quad b = 1 - a, \quad a_1 = \frac{\Omega R^2 - 1}{\ln R}, \quad c_2 = \frac{1 - \Theta}{1 - R^{\varkappa \Pr}},
$$
$$
c_1 = 1 - c_2, \quad \Theta = \frac{\Theta_2}{\Theta_1}, \quad R = \frac{R_2}{R_1}, \quad \Omega = \frac{\Omega_2}{\Omega_1},
$$

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$Ra = \beta\Theta_1 c_2 \mathrm{Pr}$ is the Rayleigh number, $\mathrm{Re} = \frac{\Omega_1 R_1^2}{\nu}$ is the Reynolds number, $\mathrm{Pr} = \frac{\nu}{\chi}$ is the Prandtl number, while ν, χ, β are, respectively, the coefficients of kinematic viscosity, thermal diffusion and thermal expansion; $\varkappa = u_1\mathrm{Re} = u_2\mathrm{Re}\,R$ is the radial Reynolds number, and u_1, u_2 are the radial velocities at $r = 1$, $r = R$, respectively. The radial flow is inward for $\varkappa < 0$ and outward for $\varkappa > 0$.

The flow with velocity vector V_0, temperature T_0, pressure Π_0 is called the main stationary flow.

Perturbations of the main flow are defined by means of the nonlinear system of equations, where $\vec{V}(v_r, v_\varphi, v_z)$, Π, T are respectively perturbation velocity, pressure and temperature components:

$$\begin{aligned}
&\frac{\partial v_r}{\partial t} + (\vec{V},\nabla)v_r + \omega_1\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi^2}{r} + \mathrm{Ra}\,\omega_2 T = \\
&\qquad = -\frac{1}{\mathrm{Re}}\frac{\partial\Pi}{\partial r} + \frac{1}{\mathrm{Re}}\Big(\Delta v_r - \frac{1-\varkappa}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial\varphi}\Big),\\
&\frac{\partial v_\varphi}{\partial t} + (\vec{V},\nabla)v_\varphi + \omega_1\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r v_\varphi}{r} - g_1(r)v_r = \\
&\qquad = \frac{1}{r\mathrm{Re}}\frac{\partial\Pi}{\partial\varphi} + \frac{1}{\mathrm{Re}}\Big(\Delta v_\varphi - \frac{1+\varkappa}{r^2}v_\varphi + \frac{2}{r^2}\frac{\partial v_r}{\partial\varphi}\Big),\\
&\frac{\partial V_z}{\partial t} + (\vec{v},\nabla)v_z + \omega_1\frac{\partial v_z}{\partial\varphi} = -\frac{1}{\mathrm{Re}}\frac{\partial\Pi}{\partial z} + \frac{1}{\mathrm{Re}}\Delta v_z,\\
&\frac{\partial T}{\partial t} + (\vec{v},\nabla)T + \omega_1\frac{\partial T}{\partial\varphi} + g_2 v_r = \frac{1}{\mathrm{Re}\,Pr}\Delta T,\\
&\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_r}{\partial z} = 0;\\
&\qquad \vec{V}|_{r=1,R} = 0, \quad T|_{r=1,R} = 0,
\end{aligned} \tag{2}$$

where $g_1(r) = \dfrac{dv_{0\varphi}}{dr} + \dfrac{v_{0\varphi}}{r}$, $\omega_1 = \dfrac{v_{0\varphi}}{r}$, $\omega_2 = \omega_1^2 r$, $g_2 = \varkappa r^{\varkappa Pr - 1}$.

The system of equations (2) is written in terms of the Boussinesq approximation, where $\beta\theta_1 \ll 1$. It is assumed that the velocity, temperature and pressure components are periodic with the known periods $2\pi/\alpha$, and $2\pi/m$, respectively.

In [2], we have considered the system of equations (2) for rotationally-symmetric (independent of φ) parameters. This system was reduced to the nonlinear operator equation which is completely continuous in the corresponding Hilbert space. To apply the bifurcation theory of nonlinear operator equations [3], it have been proved that the following conditions were fulfilled: if $\varkappa\,\mathrm{Ra} > 0$ and the functions $\omega_k(r)$, $g_k(r)$ $(k = 1, 2)$ are positive through the interval $(1, R)$, then the linear operator has at least one positive simple characteristic number Re which is the bifurcation point of the nonlinear operator (for all α, except some countable set). Concretely, for $\theta_2 > \theta_1$ (the temperature of the outer cylinder exceeds that of the inner one) and $0 < \Omega < 1/R^2$ both with outflow ($\varkappa > 0$) and inflow ($\varkappa < 0$), the main stationary flow (1) gives rise to bifurcation of the secondary stationary flow. Applying [3] and using the results of [2], we can prove the following

Theorem. *Let* $\varkappa\mathrm{Ra} > 0$, $\omega_k(r)$, $g_k(r)$ $(k = 1, 2)$ *be positive in the interval* $(1, R)$*. Then for sufficiently large Reynolds number Re, the main stationary flow* (1) *loses its stability.*

It suffices to show that for an arbitrary number Re in the spectrum of stability of the main flow (1) there appears the eigenvalue σ (perturbation decrement) with the positive real part $\sigma_r > 0$.

Consequently, in case $\theta_2 > \theta_1$ and $0 < \Omega < 1/R^2$, for any $\varkappa$ and for large Re, the main flow loses its stability.

This theorem is analogous to that for impermeable cylinders which has been proved in his book by B. Kolesov [4]. Namely, if $0 < \Omega < 1/R^2$, and the temperature of the inner cylinder does not exceed that of the outer one ($\theta_2 > \theta_1$), then for sufficiently large Reynolds number the Couette nonisothermal flow ($\varkappa = 0$) loses its stability.

The book presents the sufficient conditions for the stability, as well. For example, for $\Omega = \frac{1}{R^2}$, if the condition that the temperature of the inner cylinder exceeds that of the outer one ($\theta_1 > \theta_2$) is fulfilled, then the flow in the impermeable cylinders ($\varkappa = 0$) is stable.

In the case of permeable cylinders we can prove the following

Theorem. *If the temperature of the inner cylinder exceeds that of the outer one* ($\theta_1 > \theta_2$) *for* $\Omega = \dfrac{1}{R^2}$*, then the flow between permeable cylinders is stable with respect to rotationally-symmetric perturbations for an arbitrary Reynolds number if we have the following values of the parameters* $\varkappa$ *and* Pr*:*

$$\begin{aligned} -\frac{2}{P_r - 1} < \varkappa < -\frac{2}{P_r}, \quad P_r > 1, \\ 0 < \varkappa < \frac{2}{1 - P_r}, \quad P_r < 1. \end{aligned} \tag{3}$$

The proof of this theorem can be obtained by taking into account that $v_\varphi = 0$ in the case under consideration. It suffices to show that under the conditions (3) in the domain $D : [1, R] \times [-\pi/\alpha, \pi/\alpha]$ the functional

$$\frac{\mathrm{Re}}{2}\frac{\partial}{\partial t}\int_D (v_r^2 + v_z^2 + P_r h T^2) r dr dz = -J_1 - J_2,$$

where $h(r) = -\operatorname{Ra}\omega_2/g_2$ and

$$J_1 = -\int\limits_D \Big[\Big(\frac{\partial v_r}{\partial r}\Big)^2 + \Big(\frac{\partial v_r}{\partial z}\Big)^2 + \Big(\frac{\partial v_z}{\partial r}\Big)^2 + \frac{v_r^2}{r}\Big] r dr dz,$$

$$J_2 = -\int \Big[\Big(\frac{\partial T}{\partial r}\Big)^2 h(r) + \frac{\partial T}{\partial r} T \Big(r\frac{dh(r)}{dr} + \varkappa h(r)\Big)\Big] dr dz$$

are negative, and hence the flow is stable for any Re.

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