Proc. A. Razmadze Math. Inst. 162(2013), 111–115

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NON-SEPARABLE EXTENSIONS OF INVARIANT BOREL MEASURES AND MEASURABILITY PROPERTIES OF REAL-VALUED FUNCTIONS

One of important topics in contemporary measure theory is concerned with the problem of the existence of a nontrivial σ-finite continuous (i.e., diffused) measure on a sufficiently large class of subsets of an initial base set E, which is usually assumed to be uncountable. In general, it is impossible to define (within **ZFC** set theory) a non-zero σ-finite continuous measure on the family of all subsets of E. As a rule, for any such measure, the class of all measurable subsets of E is relatively poor. However, various methods are known of extending an original measure in order to substantially enrich its domain. Notice that proceeding in this way one can obtain even non-separable extensions of the initial separable measures. In particular, the study of non-separable extensions of Borel measures in infinite-dimensional topological groups or in topological vector spaces is of special interest. There is a rather developed methodology which allows to investigate different aspects of the above-mentioned topic (in this connection, see e. g. [1]-[7]). Here we would like to consider some types of non-separable σ-finite measures from the point of view of the concept of measurability of real-valued functions with respect to certain classes of measures.

Throughout this article, the following notation will be used:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = 2^\omega$);

ω is the cardinality of $\mathbf{N}$;

$\mathrm{dom}(\mu)$ is the domain of a given measure μ;

μ' is the completion of a given measure μ;

$\mathbf{B}(\mathbf{R})$ is the Borel σ-algebra on $\mathbf{R}$.

Among non-separable extensions of invariant measures, the most interesting and important are those which have the so-called uniqueness property (see the definition below). It is well known that the uniqueness property for

2010 *Mathematics Subject Classification.* 28A05, 28D05.

Key words and phrases. Non-separable measure, invariant measure, measurability of functions.

invariant measures plays a significant role in various questions of modern analysis and general topology. For instance, the Haar measure on a locally compact topological group has the uniqueness property and this fact implies many important consequences in abstract harmonic analysis, in the theory of dynamical systems, etc. (see, e.g., [5], [6]).

Let E be a nonempty set, G be a group of transformations of E and let μ_1 be a σ-finite G-invariant measure defined on some σ-algebra of subsets of E. We say that the measure μ_1 has the uniqueness property if, for any σ-finite G-invariant measure μ_2 defined on $\mathrm{dom}(\mu_1)$, there exists a coefficient $t \in \mathbf{R}$ (certainly, depending on μ_2) such that $\mu_2 = t \cdot \mu_1$(in other words, μ_1 and μ_2 are proportional measures).

Theorem 1. *Let (E_i, G_i, μ_i) $(i \in I)$ be an uncountable family of measurable spaces equipped with non-atomic probability G_i-invariant measures. If each measure μ_i possesses the uniqueness property, then the product measure $\prod_{i\in I} \mu_i$ is non-separable $\prod_{i\in I} G_i$-invariant and also possesses the uniqueness property.*

For the proof of Theorem 1, see e.g. [5].

Theorem 2. *Let (E_i, G_i, μ_i) $(1 \le i \le n)$ be a finite family of measurable spaces equipped with nonzero σ-finite G_i-invariant measures and let each measure μ_i be metrically transitive with respect to a countable subgroup of G_i. If at least one measure from this family is non-separable, then the product measure $\mu = \prod_{1\le i\le n} \mu_i$ is σ-finite, non-separable, $\prod_{1\le i\le n} G_i$-invariant and metrically transitive with respect to the product group $\prod_{1\le i\le n} G_i$.*

Consequently, the completion of μ is a σ-finite non-separable invariant measure having the uniqueness property.

It is possible to construct a measure on the infinite-dimensional topological vector space $\mathbf{R}^{\mathbf{N}}$, which is σ-finite, non-separable, metrically transitive and extends some invariant Borel measure χ on $\mathbf{R}^{\mathbf{N}}$ (see information on χ in [5]).

Lemma 1. *Let φ denote the first ordinal of cardinality continuum* $\mathbf{c}$. *Then in the space $\mathbf{R}^{\mathbf{N}}$ there exists a subgroup G of the group of all translations of $\mathbf{R}^{\mathbf{N}}$ and two families $(A^0_\xi : \omega \le \xi < \varphi)$ and $(A^1_\xi : \omega \le \xi < \varphi)$ of subsets of $\mathbf{R}^{\mathbf{N}}$ such that:*

1. *G is a group with basis $(x_\xi : \xi < \varphi)$;*

2. *for each closed subset F of $\mathbf{R}^{\mathbf{N}}$ which has strictly positive measure with respect to χ, we have*

$$card(F \cap \{x_\xi : \xi < \varphi\}) = 2^\omega;$$

3. *$(x_\xi : 0 \le \xi < \omega)$ is everywhere dense in* $\mathbf{R}^{\mathbf{N}}$;

4. *for each ordinal number* $\xi \in [\omega, \varphi[$, *denote by* H_ξ *a subgroup of* G *with index* 2 *such that*

$$h \in H_\xi \Rightarrow (h = \sum_{\omega \leq \xi \leq \varphi} \alpha_\xi x_\xi : \alpha_\xi \ \textit{are even integer numbers});$$

then we have the implications

$$h \in H_\xi \Rightarrow (card(A^0_\xi \triangle h(A^0_\xi)) < 2^\omega \& \ card(A^1_\xi \triangle h(A^1_\xi)) < 2^\omega)$$

and

$$h \in G \backslash H_\xi \Rightarrow (card(A^0_\xi \triangle h(A^1_\xi)) < 2^\omega \& \ card(A^1_\xi \triangle h(A^0_\xi)) < 2^\omega);$$

5. $A^0_\xi \cap A^1_\xi = \emptyset$, $\quad (\omega \leq \xi < \varphi)$;
6. $A^0_\xi \cup A^1_\xi = A^0_\zeta \bigcup A^1_\zeta \quad (\omega \leq \xi < \varphi, \omega \leq \zeta < \varphi)$;
7. *if* $(\xi_k : k \in N)$ *is an injective countable family of ordinal numbers from interval* $[\omega, \varphi[$, *then the intersection* $\cap_{k \in \mathbf{N}} A^{i_k}_{\xi_k}$ $(i_k = 0, 1)$ *is a* χ*-massive subset in* $\mathbf{R}^\mathbf{N}$.

Let us denote by $F(\mathbf{R}^\mathbf{N})$ the class of all subsets of $\mathbf{R}^\mathbf{N}$ whose cardinalities are strictly less than $\mathbf{c}$ and denote by S the σ-algebra of subsets of $\mathbf{R}^\mathbf{N}$ generated by the union

$$F(\mathbf{R}^\mathbf{N}) \cup K_\mathbf{N} \cup (\cup_{\omega \leq \xi < \varphi} \ \{A^0_\xi, A^1_\xi\}),$$

where $K_\mathbf{N}$ is the family of all χ-measurable subsets in $\mathbf{R}^\mathbf{N}$.

The next statement is valid.

Theorem 3. *On the* σ*-algebra* S *there exists a non-separable extension* $\overline{\chi}$ *of the Borel measure* χ, *which is invariant with respect to an everywhere dense vector subspace of* $\mathbf{R}^\mathbf{N}$ *and has the uniqueness property. More precisely, the weight of* $\overline{\chi}$ *is equal to* $\mathbf{c}$.

Let E be a base set and let M be a class of measures on E (we assume, in general, that the domains of measures from M are various σ-algebras of subsets of E). We shall say that a real-valued function $f : E \to \mathbf{R}$ is relatively measurable with respect to M if there exists at least one measure $\mu \in M$ such that f is measurable with respect to μ. Otherwise, we shall say that f is absolutely nonmeasurable with respect to M (see [5], [6]).

Example. Let V be an equivalence relation on $\mathbf{R}$ whose all equivalence classes are at most countable. We shall say that $f : \mathbf{R} \to \mathbf{R}$ is a Vitali type function for V if $(r, f(r)) \in V$ for each $r \in \mathbf{R}$ and the set ran(f) is a selector of the partition of $\mathbf{R}$ determined by V. Let M_1 be the class of all translation invariant extensions of the Lebesgue measure λ on $\mathbf{R}$ and let M_2 be the class of all translation quasi-invariant extensions of λ on $\mathbf{R}$. Then there exists a Vitali type function which is relatively measurable with respect to the class M_2 and is absolutely nonmeasurable with respect to the class M_1.

The above-mentioned example is discussed in [5] and [6].

The next two results are well known in measure theory.

Lemma 2. *Let (E, S) be a measurable space. Then the following two assertions are equivalent:*

1. *S is a countably generated σ-algebra of subsets of E;*
2. *there exists a function $f : E \to \mathbf{R}$ such that*

$$S = \{f^{-1}(B) : B \in \mathbf{B}(\mathbf{R})\}.$$

Lemma 3. *If S is a countably generated σ-algebra on $\mathbf{R}$, then any σ-finite measure μ defined on S is separable.*

Remark. Notice that the converse assertion to Lemma 3 is not valid.

Let L_1 be the class of all nonzero σ-finite separable measures on $\mathbf{R}$ and let L_2 be the class of all nonzero σ-finite non-separable measures on $\mathbf{R}$.

According to the above-mentioned lemmas, we have the next statement.

Theorem 4. *If a function $f : E \to \mathbf{R}$ is relatively measurable with respect to the class L_2, then f is relatively measurable with respect to the class L_1.*

Acknowledgement

The research has been partially supported by Shota Rustaveli National Science Foundation, Grant No. 31/25.

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