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INTEGRATION OF DOUBLE FOURIER TRIGONOMETRIC SERIES

There is the well-known Lebesgue theorem stating that for a Fourier series $f \sim \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, even if the later is everywhere divergent, the following equalities are fulfilled for all x:

$$\int\limits_0^x f(t)\,dt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \int\limits_0^x (a_n \cos nt + b_n \sin nt)\,dt,$$

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \frac{1}{\pi} \int\limits_0^{2\pi} f(x)\,\frac{1}{2}\,(\pi - x)\,dx.$$

The main result is formulated as follows:

Theorem. *Let a function f be summable on the square $[0, 2\pi]^2$, be 2π-periodic with respect to each independent variable and*

$$\begin{aligned} f \sim \frac{1}{4}\,a_{00} + \frac{1}{2} \sum_{m=1}^{\infty} (a_{m0} \cos mx + d_{m0} \sin mx) + \\ + \frac{1}{2} \sum_{n=1}^{\infty} (a_{0n} \cos ny + c_{0n} \sin ny) + \\ + \sum_{m,n=1}^{\infty} \big(a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny + \\ + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \cos ny\big). \end{aligned}$$

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Then the following equalities are valid:

$$1)\quad \int_0^x\int_0^y f(t,\tau)\,dt\,d\tau=\frac{1}{4}\,a_{00}xy+\frac{1}{2}\,y\sum_{m=1}^{\infty}\int_0^x(a_{m0}\cos mt+d_{m0}\sin mt)\,dt+$$

$$+\frac{1}{2}\,x\sum_{n=1}^{\infty}\int_0^y(a_{0n}\cos n\tau+c_{0n}\sin n\tau)\,d\tau+$$

$$+\sum_{m,n=1}^{\infty}\int_0^x\int_0^y\bigl[a_{mn}\cos mt\cos n\tau+b_{mn}\sin mt\sin n\tau+$$

$$+c_{mn}\cos mt\sin n\tau+d_{mn}\sin mt\cos n\tau\bigr]\,dt\,d\tau;$$

$$2)\quad \sum_{m,n=1}^{\infty}\frac{1}{mn}\,b_{mn}=-\frac{1}{4}\,A_{00}-\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,\beta_m+\frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,d_{m0}-$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,\delta_n+\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,c_{0n},$$

where

$$A_{00}=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}F(x,y)\,dx\,dy,$$

$$F(x,y)=\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau-y\,\frac{1}{2\pi}\int_0^x dt\int_0^{2\pi}f(t,\tau)\,d\tau-$$

$$-x\,\frac{1}{2\pi}\int_0^y d\tau\int_0^{2\pi}f(t,\tau)\,dt,$$

$$\beta_m=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}yf(x,y)\sin mx\,dx\,dy,$$

$$\delta_n=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}xf(x,y)\sin ny\,dx\,dy.$$

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