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# THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS FROM VARIABLE EXPONENT SMIRNOV CLASSES IN DOMAINS WITH A PIECEWISE SMOOTH BOUNDARY

When boundary value problems of the theory of analytic functions are unsolvable in classes of smooth functions, it is advisable to solve them in Smirnov classes of analytic functions $E^p(D)$. Functions of these classes possess angular boundary functions belonging to the Lebesgue class $L^p$, hence these problems can be considered in the cases where the functions given in the boundary conditions are unbounded. Moreover, since for $p \geq 1$ the functions of the class $E^p(D)$ are representable by the Cauchy integral, one frequently succeeds in constructing solutions in quadratures. According to the above-said, the boundary value problems for harmonic functions (in particular, the Dirichlet problem, as well) can naturally be considered in the classes $e^p(D) = \{u : u = \operatorname{Re}\Phi,\ \Phi \in E^p(D)\}$. Such a setting is the more so convenient in that the functions of these classes, being the real parts of the Cauchy integrals, are a combination of simple- and double-layer potentials (see [1], §12).

The boundary value problems in the classes mentioned above were solved by many authors (see, for e.g., [2]).

Recently, when solving various problems of applied character, in the role of unknowns were used sets of functions possessing boundary values which form Lebesgue integrable functions with a variable exponent $p(t)$. The spaces $L^{p(t)}$ allow one to pay more attention to local singularities of the given boundary functions and thus to find solutions from more natural subclasses of summable functions.

In this connection, there naturally arose the question on introducing such Smirnov classes with a variable exponent $p(t)$ which, maintaining inherent in them basic properties for a constant exponent, possess boundary functions from $L^{p(t)}$. Such classes have been introduced in [3].

**The Classes $E^{p(t)}(D,\omega)$.** Let $D$ be a simply-connected bounded domain with the boundary $\Gamma$, where $\Gamma$ is a simple rectifiable curve whose equation with respect to the arc abscissa $s$ is $t = t(s)$, $0 \leq s \leq l$, and $z = z(\omega)$ is

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conformal mapping of the unit circle $U = \{w : |w| < 1\}$ with the boundary $\gamma = \{t : |t| = 1\}$ onto $D$. Moreover, let $\omega = \omega(w)$ be a measurable, almost everywhere distinct from zero in $D$, and $p = p(t)$ be a positive measurable function on $\Gamma$.

We say that the analytic in $D$ function $\Phi$ belongs to the class $E^{p(t)}(D, w)$ if

$$\sup_{0<r<1} \int_0^{2\pi} \left|\Phi(z(re^{i\vartheta}))\, \omega(z(re^{i\vartheta}))\right|^{p(z(e^{i\vartheta}))} d\vartheta < \infty.$$

The belonging of the function $\Phi$ to the class $E^{p(t)}(D, \omega)$ does not depend on the choice of the function $z(w)$. If $\omega(w)$ possesses angular boundary function $\omega^+(t)$, then $\Phi(z)$ for almost all $t \in \Gamma$ has angular boundary value $\Phi^+(t)$, and the function $\Phi^+$ belongs to $L^{p(t)}(\Gamma; \omega)$, i.e.,

$$\left\|\Phi^+\right\|_{L^{p(t)}(\Gamma;\omega)} = \inf\left\{\lambda : \int_0^{\ell} \left|\frac{\Phi^+(t(s))\omega^+(t(s))}{\lambda}\right|^{p(t(s))} ds \le 1\right\} < \infty.$$

Assume $E^{p(t)}(D) = E^{p(t)}(D; 1)$, $H^{p(t)} = E^{p(t)}(U)$, $e^{p(t)}(D; \omega) = \{u : u = \operatorname{Re}\Phi,\ \Phi \in E^{p(t)}(D; \omega)\}$, $e^{p(t)}(D) = e^{p(t)}(D; 1)$, $h^{p(t)}(\omega) = e^{p(t)}(U; \omega)$.

**The Classes of Exponents.** Let $\varepsilon \ge 0$. The function $p : p \to \mathbb{C}$ belongs to the class $\mathcal{P}_{1+\varepsilon}(\Gamma)$, if: (a) $\inf\limits_{t\in\Gamma} p(t) > 1$; (b) there exists a number $B$, such that for any $t_1, t_2 \in \Gamma$ we have $|p(t_1) - p(t_2)| < B|\ln|t_1 - t_2||^{-(1+\varepsilon)}$.

Assume $\widetilde{\mathcal{P}}(\Gamma) = \bigcup\limits_{\varepsilon>0} \mathcal{P}_{1+\varepsilon}(\Gamma)$.

Let $z = z(w)$ be conformal mapping of the unit circle $U$ onto $D$. We say that $p \in Q(\Gamma)$, if $p \in \widetilde{\mathcal{P}}(\Gamma)$ and the function $\ell(\tau) = p(z(\tau))$ belongs to $\widetilde{\mathcal{P}}(\gamma)$.

By $C^1_D(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$ we denote a set of piecewise smooth simple curves $\Gamma$ with angular points $t_k$, $k = \overline{1, n}$ at which the values of angles, interior with respect to $D$, are equal to $\pi\nu_k$, $0 \le \nu_k \le 2$.

If $z = z(w)$ is conformal mapping of the circle $U$ onto $D$ with the boundary $\Gamma \in C^1_D(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$, then

$$z'(w) \sim \prod_{k=1}^{n} (w - \tau_k)^{\nu_k - 1} \exp\left(\int_\gamma \frac{\psi(\tau)d\tau}{\tau - w}\right), \tag{1}$$

where $z(\tau_k) = t_k$, and $\psi$ is the real, continuous on $\gamma$ function, and the writing $f \sim g$ denotes that $0 < \inf\left|\frac{f}{g}\right| \le \sup\left|\frac{f}{g}\right| < \infty$ (see [4], p. 144).

**Statement of the Problem.** Let $D$ be the simply-connected bounded domain with the boundary $\Gamma \in C^1_D(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$, and $p \in Q(\Gamma)$. Find the function $u$ for which

$$\begin{cases} \Delta u = 0, & u \in e^{p(t)}(D), \\ u^+(t) = f(t), & t \in \Gamma, \quad f \in L^{p(t)}(\Gamma). \end{cases} \tag{$\mathcal{D}$}$$

**Theorem.** *Let* $\Gamma \in C^1_D(t_1, \nu_1)$, $p \in Q(\Gamma)$, $\ell(\tau) = p(z(\tau))$, $\tau \in \Gamma$, *and* $w = w(z)$ *be conformal mapping of* $D$ *onto* $U$, *then:*

I. *If* $0 < \nu_1 < p(t_1)$, *the problem* $(\mathcal{D})$ *is uniquely solvable.*

II. *If* $\nu_1 > p(t_1)$, *the problem is solvable, and its general solution contains one arbitrary real constant.*

III. *If* $\nu_1 = p(t_1)$, *the problem* $(\mathcal{D})$ *is solvable only for those* $f$ *for which*

$$\omega_1(\tau) \int\limits_\gamma \frac{f(z(\tau))}{\omega_1(\zeta)} \frac{d\zeta}{\zeta - \tau} \in L^{\ell(\cdot)}(\gamma), \tag{2}$$

*where*

$$\omega_1(\tau) = \omega_1^+(\tau), \quad \omega_1(w) = (w - \tau_1)^{-\frac{\nu_1}{p(t_1)}} \exp \int\limits_\gamma \frac{\psi(\zeta)}{\ell(\zeta)} \frac{d\zeta}{\zeta - w}$$

*and* $\psi$ *is the function appearing in* (1). *If the condition* (2) *is fulfilled, then general solution of the homogeneous problem is given by the equality*

$$u_0(z) = M(p)[w(z) + t_1][w(z) - t_1]^{-1}, \tag{3}$$

*where*

$$M(p) = \begin{cases} 0, \quad \text{if} \quad \omega_1(w) \overline{\in} H^{\ell(\cdot)}, \\ \text{is an arbitrary constant, if} \quad \omega_1 \in H^{\ell(\cdot)}. \end{cases} \tag{4}$$

IV. *If* $\nu_1 = 0$, *then the problem* $(\mathcal{D})$ *is solvable, iff the condition* (2) *is fulfilled, and if it is fulfilled, the problem is uniquely solvable.*

*In all cases where the problem is solvable, the solution is given by the equality*

$$u(z) = \operatorname{Re}\left( \frac{\omega_1(w(z))}{2\pi i} \int\limits_\gamma \frac{f(z(\zeta)}{\omega_1^+(\zeta)} \frac{d\zeta}{\zeta - w(z)} \right) + u_0(z),$$

*where* $u_0(z)$ *is defined by* (3) *and* (4).

*In a general case for the curve* $\Gamma \in C^1_D(t_1, \dots, t_n; v_1, \dots, v_n)$, *analogous results are valid. In particular, a general solution contains at least as many arbitrary constants as there exist angular points* $t_k$ *at which* $\nu_k > p(t_k)$ (*there may appear additional solutions for some* $t_k$ *at which* $\nu_k = p(t_k)$).

*Remarks.* (1) If $\Gamma$ is a piecewise Lyapunov curve and $\nu_1 = p(t_1)$, then the problem is uniquely solvable.

(2) If $\Gamma$ is a smooth curve, then the problem $(\mathcal{D})$ is unconditionally solvable, and the solution is unique.

(3) If $p(t) = p = \text{const} > 1$, then the condition of solvability (2) is fulfilled, iff $f(t) \ln |w(t) \to t_1| \in L^p(\Gamma)$ (see [4], p. 168).

When investigating the problem $(D)$, we have to a considerable extent used the results presented in [1]–[6]. We will especially dwell here on the following fact.

Using conformal mapping, we reduce the problem $\mathcal{D}$ to the problem $(\mathcal{D})$ for the circle in the class $h^{\ell(\tau)}(\omega)$, with weight $\omega$ of somewhat complicated type. We state that $\omega(\tau) \sim \omega_1(\tau)$. Then the problem $(\mathcal{D})$ in $h^{p(\tau)}(\omega_1)$, following the method of N. Muskheishvili (used by him to reduce the Riemann-Hilbert problem to the Riemann problem), we reduce to the Riemann problem in the class $\widetilde{K}^{\ell(\cdot)}(\gamma)$ (of Cauchy type integrals with density from $L^{\ell(\cdot)}(\gamma)$ and with the constant principal part at infinity). The investigation of the latter problem leads to the results mentioned above.


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