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SOME NONLOCAL PROBLEMS FOR HIGHER ORDER SINGULAR ORDINARY DIFFERENTIAL EQUATIONS

On the finite interval $[a,b]$, the nonlinear differential equation

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

with nonlocal conditions

$$\int_a^b u^{(k-1)}(t)d\varphi_k(t) = 0 \quad (k = 1, \dots, n) \tag{2}$$

is considered. Here $f : [a,b] \times R_0^n \to R$ is a continuous function,

$$R_0^n = \big\{(x_1, \dots, x_n) : x_1 \neq 0, \dots, x_n \neq 0, \big\},$$

and $\varphi_i : [a,b] \to R$ $(i = 1, \dots, n)$ are functions with bounded variations such that

$$\varphi_k(0) = 0, \ \ \varphi_k(1) = 1, \ \ \sigma_k \varphi_k(t) > \frac{1+\sigma_k}{2} \ \text{ for } \ a < t < b \ \ (k = 1, \dots, n),$$

where $\sigma_k \in \{-1, 1\}$ $(k = 1, \dots, n)$.

We are mainly interested in the case where Eq. (1) is singular, precisely, the cases when there does not exist the finite limit

$$\lim_{\sum\limits_{k=1}^{n} |x_k| \to 0} f(t, x_1, \dots, x_n)$$

for some $t \in (a,b)$.

Suppose $R_+ = [0, +\infty[$,

$$R_{0+}^n = \big\{(x_1, \dots, x_n) : x_1 > 0, \dots, x_n > 0\big\}, \quad \sigma_{0k} = \prod_{i=k}^{n} \sigma_i \ \ (k = 1, \dots, n),$$

$$\ell_k = \prod_{i=k}^{n-1} \int_a^b \Big|\varphi_i(t) - \frac{1-\sigma_i}{2}\Big| dt \ \ (k = 1, \dots, n-1), \quad \ell_n = 1.$$

The following theorem is valid.

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Theorem. *Let the inequality*

$$q_0(t, |x_1|, \dots, |x_n|) \le f(t, x_1, \dots, x_n) \le \sum_{k=1}^{n} p_k(t)|x_k| + q(t, |x_1|, \dots, |x_n|)$$

be fulfilled on the set $[a, b] \times R_0^n$, *where* $p_k : [a, b] \to R_+$ $(k = 1, \dots, n)$, q_0 *and* $q : [a, b] \times R_{0+}^n \to R_+$ *are continuous functions, at that* q_0 *and* q *are, respectively, non-decreasing and non-increasing in the last* n *arguments. Let, moreover,*

$$\lim_{x>0, x\to 0} \int_a^b \frac{q_0(t, x, \dots, x)}{x} dt = +\infty$$

and

$$\sum_{k=1}^{n} \ell_k \int_a^b \Big|\varphi_n(t) - \frac{1-\sigma_n}{2}\Big| h_k(t) dt \le 1.$$

Then problem (1), (2) *has at least one solution, satisfying the inequalities*

$$\sigma_{0k} u^{(k-1)}(t) > 0 \ \ for \ \ a \le t \le b \ \ (k = 1, \dots, n). \tag{3}$$

For the differential equation

$$u^{(n)} = \sum_{k=1}^{n} \Big(f_{1k}(t)|u^{(k-1)}|^{\lambda_k} + f_{2k}(t)|u^{(k-1)}|^{-\mu_k} \Big), \tag{4}$$

from the above-formulated theorem it follows

Corollary. *Let*

$$0 < \lambda_k < 1, \quad \mu_k > 0 \ \ (k = 1, \dots, n),$$

functions $f_{ik} : [a, b] \to R_+$ $(i = 1, 2; k = 1, \dots, n)$ *be continuous and*

$$\sum_{k=1}^{n} f_{1k}(t) \not\equiv 0.$$

Then problem (4), (2) *has at least one solution, satisfying inequalities* (3).

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