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EXTREMAL PROBLEMS ON MODULI SPACES OF MECHANICAL LINKAGES

Mechanical linkages are widely used as models of various mechanisms and physical systems [5] and the study of their equilibria for various potential functions leads to extremal problems on configuration spaces of such linkages [2], [7]. Motivated by such problems, we studied the critical points of several concrete functions on configuration spaces of mechanical linkages using methods of real algebraic geometry [4] and singularity theory [1]. This enabled us to develop a new point of view on several well-known topics and obtain a number of new results, which suggested two general paradigms concerned with the critical points of regular functions on configuration spaces of linkages. In the sequel, we shall present some of the aforementioned new results and formulate two paradigms. It should be added that our approach and results have been developed by several Georgian and foreign authors [3], [6], [8], [12], [13], [14].

Recall that linkages are defined as mechanisms build up from rigid bars (sticks) joined at flexible links (pin-joints). In many problems it is important to know the totality of possible positions of the links in the ambient space, which led to a mathematical definition of moduli space of a mechanical linkage discussed in big detail in [5], [9]. Moduli spaces are often called the configuration spaces of linkages, but we will only use the term moduli space.

For simplicity, we only consider *planar* moduli spaces of *chain linkages*. A chain linkage L is defined by a sequence $l = (l_1, \dots, l_n)$ of positive numbers l_i called the *code* (or *sidelength vector*) of L. A planar configuration (realization) of a chain linkage L is defined as the sequence (chain) of consecutive straight line segments in Euclidean plane $\mathbb{R}^2$ such that the length of ith segment is equal to l_i. A configuration can be equivalently defined by its set of vertices $V = (v_1, \dots, v_n)$. If we require that $v_n = v_1$ this corresponds to a *planar polygonal linkage*. If such a condition is not imposed, then we speak of a *planar robot arm* (or *planar multiple pendulum*).

In both cases the *planar moduli space* $\mathcal{M}_2(L)$ of L is defined as the factor (orbit-space) of the set $V(L)$ of all realizations of L in $\mathbb{R}^2$ modulo the

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diagonal action of the group $Iso_+(\mathbb{R}^2)$ of orientation preserving isometries of $\mathbb{R}^2$. For an n-arm R, $\mathcal{M}_2(R)$ is obviously homeomorphic to $(n-1)$-torus T^{n-1}, while for a n-gonal linkage it has a natural structure of a $(n-3)$-dimensional compact algebraic variety which is nonsingular (actually, smooth) for generic code l [5]. Thus, for generic l, $\mathcal{M}_2(L)$ is a smooth manifold and one can consider smooth (differentiable) functions on it. The strategy of our research is to construct various Morse functions on moduli spaces and study their critical points.

We illustrate this general strategy by considering the oriented area of planar polygon as a function on the planar moduli space of a chain linkage. Recall that given an ordered set of n points $v_1 = (x_1, y_1), \dots, v_n = (x_n, y_n)$ in the plane, the oriented area A of the corresponding n-gon is defined by the formula

$$2A = (x_1y_2 - y_1x_2) + \cdots + (x_ny_1 - y_nx_1).$$

This obviously gives a differentiable (smooth) function $A_L : \mathcal{M}_2(L) \to \mathbb{R}$ for each n-chain linkage. If $\mathcal{M}_2(L)$ is smooth then one can consider the critical points of A_L and our main aim is to obtain geometrical and topological information about the critical configurations of A_L. Based on the results obtained for concrete classes of linkages [10], [6], the following four general conjectures have been formulated in [11].

Conjectures CA. *For a generic n-gon linkage L with smooth planar configuration space $\mathcal{M}_2(L)$, the following four statements hold true:*

$(CA1)$ *the critical points of area on $\mathcal{M}_2(L)$ are given by cyclic configurations of L;*

$(CA2)$ *the critical values of area can be calculated as the roots of a certain explicitly constructible polynomial;*

$(CA3)$ *the critical points are non-degenerate in the sense of Morse theory;*

$(CA4)$ *the Morse indices of cyclic configurations can be read off their shape.*

These conjectures obviously make sense for a planar robot arm, in which setting they will be referred to as (CA1*,...,CA4*). It should also be noted that there is no direct reduction of one setting to another and in fact there are essential differences between these two settings.

All these conjectures appeared to be true (some under additional assumptions) and we now describe the main results related to conjectures (CA1-CA4). We believe that they may serve as a paradigm for studying the critical points of other geometrically or physically meaningful functions on moduli spaces like Coulomb energy of unit charges placed at vertices [7], sum of lengths of diagonals or normalized determinant considered by M. Atiyah [2]. It should be noted that (CA2) is equivalent to a weakened form of conjecture formulated by D. Robbins in [15] as a statement concerned with calculation of the areas of cyclic polygons in terms of the lengths of their sides. However D. Robbins did not use the concepts of linkage and moduli space so the connection between the two settings could only be established after having proven (CA1).

Conjecture (CA1) has been proven [12] for arbitrary n under mild additional assumptions. Thus our proof of (CA1) automatically implies validity of (CA2) for generic polygonal linkages with arbitrary number of sides. (CA1*) has been proven for generic n-arms with n arbitrary in [13].

The third pair of conjectures appeared harder. (CA3*) has been proven in [13] for certain robot arms using the *parametric transversality* theorem described, e.g., in [1]. Situation with (CA4) and (CA4*) is more complicated and we do not discuss it here. More precisely, the following two basic results were obtained in [12] and [13], respectively.

Theorem 1 ([12])**.** *For a generic n-gon linkage L with nonsingular planar moduli space, all critical points of A on $\mathcal{M}_2(L)$ are given by the cyclic configurations of L.*

In order to formulate an analogous result for planar (robot) n-arms (or multiple planar penduli [13]) we need an "ad hoc" definition. For each configuration $(v_1, \ldots, v_n)$ of a planar n-arm define the *connecting side* as the segment $v_n v_0$. A cyclic configuration of a planar n-arm is called *diacyclic* if the center of its circumscribed circle lies on the connecting side (thus $v_n v_0$ is a diameter of the circumscribed circle).

Theorem 2. ([13]) *For a generic planar n-arm R, all critical points of A on $\mathcal{M}_2(R)$ are given by the diacyclic configurations of L.*

Both these theorems were proved by geometric methods but one can also reformulate them in purely algebraic terms, which permits to extend them to singular moduli spaces. It is easy to see that the critical points of A in moduli space $\mathcal{M}_2(n)$ can be considered as the real solutions to a certain $(2n-4) \times (2n-4)$-system S_l of polynomial equations depending on parameters l_i. Indeed, according to Lagrange rule the gradient ∇A at a critical point should be linearly dependent with the gradients of defining quadratic equations $g_i = l_i^2, i = 1, \ldots, n-1$. In other words, the rank of Jacobi matrix $(\nabla g_1, \ldots, \nabla g_{n-1}, \nabla A)^T$ should be equal to $n-1$, which is equivalent to vanishing of all of its $(n \times n)$-minors. Since the number of variables is $2n-4$, generically this can be expressed by vanishing of any collection of $n-3$ such minors. Joining the arising $n-3$ polynomial equations to the defining equations $g_i = l_i^2$ we obtain the system S_l mentioned above. Analogously, using the well-known determinantal criterion of concyclicity for four points (see, e.g., [13]) it is also easy to see that the cyclic configurations correspond to the roots of another $(2n-4) \times (2n-4)$-system T_l of polynomial equations in the same $2n-4$ unknowns. We can now define the two ideals $I(S_l)$ and $I(T_l)$ in $\mathbb{R}_{2n-4}$ as the appropriate Fitting ideals [4] of the two systems introduced above.

Theorem 3. *For arbitrary n, the ideals $I(S_l)$ and $I(T_l)$ coincide.*

This is an extension of Theorem 1 applicable in the case of a singular moduli space. We have also obtained a similar extension of Theorem 2.

The results of D.Robbins [15] combined with our Theorem 1 imply that the set of critical values of signed area coincides with the set of real roots of a certain explicitly computable polynomial. We have found a natural generalization of this fact which we formulate as a paradigm because of its generality and wide applicability. We believe that this makes sense since this formulation gives a sort of "raison d'être" for the results of [15].

Paradigm 1. *Let $f, g_1, \ldots, g_k \in \mathbb{R}_n, k \leq n-1$ be a generic set of real polynomials in n variables. Suppose that the level set $X = \{g_1 = 0, \ldots, g_k = 0\}$ is smooth and compact. Then the critical values of restriction $f|X$ are the real roots of a real polynomial in one variable whose coefficients can be algebraically expressed through coefficients of $f, g_1, \ldots, g_k$.*

For certain robot arms, (CA3*) was proven in [13] using the parametric transversality theorem [1]. The method used in [13] suggested the following general statement which we again formulate in the form of a paradigm.

Paradigm 2. *Let $f, g_1, \ldots, g_k \in \mathbb{R}_n, k \leq n-1$ be algebraically independent real polynomials in n variables such that $g_1, \ldots, g_k$ define a propomap $G : \mathbb{R}^n \to \mathbb{R}^k$ which is generically of maximal rank k. Then, for generic $l = (l_1, \ldots, l_k) \in \mathbb{R}^k$, the level surface $X_l = \{g_1 = l_1, \ldots, g_k = l_k\}$ is smooth and all critical points of restriction $f|X_l$ are nondegenerate* (*in the sense of Morse theory*).

Both these paradigms have natural extensions to the case of rational function on moduli space which make them applicable to the critical points of Coulomb energy.

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