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SOME DISCRETE GEOMETRIC STRUCTURES AND ASSOCIATED ALGORITHMS

A system Z of points in the Euclidean space $\mathbf{R}^m$ is discrete if every ball in $\mathbf{R}^m$ contains only finitely many points from Z. Such systems can be frequently met in various fields of mathematics and in applied scientific disciplines (e.g., the vertices of mosaics, crystals, networks, etc.). Their well-known objects are the so-called Delaunay systems (see [3] in which such systems were introduced for the first time; see also [13], [14] for a more detailed account). But, undoubtedly, the most important are finite subsets of points of the space $\mathbf{R}^m$, because they generate simplicial complexes and polyhedra in $\mathbf{R}^m$.

Some geometric images are naturally associated with a given finite system Z of points of $\mathbf{R}^m$. For example, one can consider the convex hull $\mathrm{conv}(Z)$ of Z which is uniquely determined by Z and is a convex polyhedron whose dimension does not exceed m (see, e.g., [6]). In general, the vertices of $\mathrm{conv}(Z)$ constitute a proper subset of Z. So, there arises the following question:

Does any finite system of points in $\mathbf{R}^m$, not contained in an affine hyperplane of $\mathbf{R}^m$, determine at least one polyhedral hypersurface which is homeomorphic to the unit sphere $\mathbf{S}_{m-1}$ and whose set of vertices coincides with this system?

The above-posed question may be regarded as a typical one for discrete, combinatorial or computational geometry and will be envisaged below. More precisely, it will be shown that any finite system of points of Euclidean space, not contained in an affine hyperplane of this space, induces a simple polyhedron of some special type. Several related questions of discrete, combinatorial and computational geometry will be touched upon, too.

First, let us recall the definitions of those geometric objects which will be exploited in the sequel.

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Consider any polyhedral hypersurface L in $\mathbf{R}^m$ homeomorphic to $\mathbf{S}_{m-1}$. Let Z be the set of all vertices of L. We say that Z determines (or induces) L and, respectively, denote $L = L(Z)$.

An m-dimensional polyhedron P in $\mathbf{R}^m$ is simple if it is homeomorphic to the m-dimensional unit ball $\mathbf{B}_m$ in $\mathbf{R}^m$. Obviously, the set $V = V(P)$ of all vertices of P determines the boundary of P (in the sense of the above definition). In this case, we will also say that V determines the simple polyhedron P.

It is easy to see that, in general, a finite set $Z \subset \mathbf{R}^m$ may determine (induce) different polyhedral hypersurfaces homeomorphic to $\mathbf{S}_{m-1}$ and different simple polyhedra. Also, it is clear that if Z lies in an affine hyperplane of $\mathbf{R}^m$, then no polyhedral hypersurface homeomorphic to $\mathbf{S}_{m-1}$ can be induced by Z, because no hyperplane contains a homeomorphic image of $\mathbf{S}_{m-1}$ (this fact is a direct consequence of the well-known Borsuk-Ulam theorem on antipodes). So, only that case is of interest when Z does not lie in an affine hyperplane of $\mathbf{R}^m$.

Let us consider briefly the easiest two-dimensional case (i.e., $m = 2$), where a finite set Z of points is given on the Euclidean plane $\mathbf{R}^2$ and satisfies the inequality $\mathrm{card}(Z) \geq 3$. Assume, for simplicity, that Z is in general position, i.e., no three points of Z are collinear. In this case, the following geometric construction (geometric algorithm) leads to the required polygonal curve induced by Z and homeomorphic to $\mathbf{S}_1$. At the first step we construct any closed polygonal curve L whose vertices are all the points of Z, but without their repeating. If L is homeomorphic to $\mathbf{S}_1$, then we are done. Otherwise, there are four points x, y, z, t in Z such that the line segments $[x, y]$ and $[z, t]$ have nonempty intersection. The curve L can be written as

$$L = xyAztB \quad (\text{or as } L = xyAtzB),$$

where A and B are some polygonal arcs. At the second step we replace L by

$$L' = xz(-A)ytB \quad (\text{or by } L' = xt(-A)yzB).$$

Of course, the symbol $-A$ here denotes the same arc A endowed with the opposite orientation. It can readily be checked that, in both above cases, the length of the obtained new polygonal curve L' is strictly less than the length of L. So, iterating this procedure sufficiently many times and taking into account the finiteness of Z, we necessarily come to a polygonal curve, homeomorphic to $\mathbf{S}_1$, whose vertices are the points of Z.

As far as we know, the above very simple algorithm was first described by P. Erdös. However, this algorithm has some weak sides. For example, one of its main defects is that it may fail to work for those finite subsets Z of $\mathbf{R}^2$ which contain four collinear points.

A much more complicated situation is in the case of finite point sets in a multi-dimensional Euclidean space. So far, no reasonable analogue of the Erdös algorithm was presented for finite point systems in $\mathbf{R}^m$. Nevertheless, it was demonstrated in [9] that if $m \geq 2$, then any finite point set Z in $\mathbf{R}^m$ not contained in an affine hyperplane of $\mathbf{R}^m$ determines a polyhedral hypersurface homeomorphic to $\mathbf{S}_{m-1}$ and the simple polyhedron associated with this hypersurface can be "triangulated" by simplices whose all vertices belong to Z. Later on, the same topic was discussed by other authors and different constructions of induced polyhedral hypersurfaces and induced simple polyhedra were presented (see, for instance, [7], [8]).

We have already said that the question of the existence of a corresponding construction of an induced polyhedral hypersurface may be included in the list of typical problems of combinatorial or computational geometry, such as:

(a) finding an optimal algorithm of triangulation of a simple polygon in $\mathbf{R}^2$;

(b) finding the minimum (nonzero) distance between points of a given finite subset of $\mathbf{R}^m$;

(c) finding an optimal algorithm for computation of the convex hull of a given finite point set in $\mathbf{R}^m$.

Detailed discussion of problems similar to (a), (b) and (c) is presented in many sources (see, for instance, [12], [13], [14]). But, as is mentioned in [8], the construction of a polyhedral hypersurface induced by a given finite point system in $\mathbf{R}^m$ was not as thoroughly considered as the problems (a), (b) and (c) indicated above. Moreover, the authors of [8] underline that, "surprisingly, this problem has received little attention".

Here we do not intend to envisage computational aspects of the problem, which involve finding corresponding optimal algorithms or optimal constructions (they are rather technical and of less interest from the purely theoretical view-point, but are very important for applications in practice). Our main goal is to show that the problem is tightly connected with a special type of polyhedra in $\mathbf{R}^m$. Recall that, according to the result obtained in [9], if a natural number $m \geq 2$ is given, then every finite subset of $\mathbf{R}^m$ not contained in an affine hyperplane of $\mathbf{R}^m$, determines a simple polyhedron (consequently, induces a polyhedral hypersurface homeomorphic to $\mathbf{S}_{m-1}$). The method developed in [9] allows one to establish a more general result indicating to a close relationship between this problem and a certain proper subclass of the class of all simple polyhedra in $\mathbf{R}^m$. To formulate the generalized result, let us first introduce the desired type of polyhedra (cf. [11]).

Let Q and Q^* be two m-dimensional polyhedra in the Euclidean space $\mathbf{R}^m$. We shall say that Q^* is an admissible extension of Q if there exists an m-dimensional simplex T in $\mathbf{R}^m$ satisfying the following two conditions:

(i) $Q \cap T$ is a common facet of Q and T;

(ii) $Q^* = Q \cup T$.

In other words, Q^* is an admissible extension of Q if and only if Q^* can be obtained by adding to Q an m-dimensional simplex T built over some facet of Q and lying outside of Q (obviously, this facet of Q should be an $(m-1)$-dimensional simplex in $\mathbf{R}^m$).

It immediately follows from the definition that if an initial Q is a simple polyhedron, then Q^* is a simple polyhedron, too.

We shall say that a finite sequence $\{Q_0, Q_1, \dots, Q_k\}$ of polyhedra in $\mathbf{R}^m$ is admissible if Q_0 is an m-dimensional simplex and, for any natural index $j \in \{0, 1, \dots, k-1\}$, the polyhedron Q_{j+1} is an admissible extension of Q_j.

Finally, we shall say that a polyhedron Q is admissible if Q is a member of some admissible sequence $\{Q_0, Q_1, \dots, Q_k\}$ of polyhedra in $\mathbf{R}^m$.

The first member Q_0 of this sequence will be called a starting simplex for Q.

One can verify (by easy induction on k) that every admissible polyhedron Q is simple and that all facets of Q are $(m-1)$-dimensional simplices.

Also, it is not difficult to see that if an m-dimensional polyhedron P is admissible and $v(P)$ denotes the number of all vertices of P, then there exists a "triangulation" of P into m-dimensional simplices whose number equals $v(P) - m$ and all whose vertices are vertices of P. In this context, it should be recalled that, for $m \geq 3$, there are simple polyhedra in $\mathbf{R}^m$ which have the property that any their "triangulation" necessarily needs additional vertices (see, for example, [1]).

In view of the above-said, the natural question arises whether any finite point set in $\mathbf{R}^m$, which does not lie in an affine hyperplane of $\mathbf{R}^m$, determines an admissible polyhedron. It turns out that the answer to this question is positive. To establish this fact, we use the following auxiliary proposition.

Lemma. *Let $[x_0, x_1, \dots, x_m]$ be a simplex in $\mathbf{R}^m$ with vertices $x_0, x_1, \dots,$ x_m, and suppose that finite sets*

$$X_1 \subset [x_0, x_1], \quad X_2 \subset [x_0, x_1], \dots, \quad X_m \subset [x_0, x_m]$$

are given. Then there exists an admissible polyhedron Q induced by the set

$$\{x_0, x_1, \dots, x_m\} \cup X_1 \cup X_2 \cup \dots \cup X_m.$$

Moreover, a starting simplex Q_0 for Q can be taken of the form

$$Q_0 = [x_0, y_1, y_2, \dots, y_m],$$

where

$$y_1 \in X_1 \cup \{x_1\}, \quad y_2 \in X_2 \cup \{x_2\}, \dots, \quad y_m \in X_m \cup \{x_m\}.$$

To show the validity of Lemma, it suffices to argue by induction on k, where

$$k = \mathrm{card}(X_1 \cup X_2 \cup \dots \cup X_m).$$

In addition, for any index $i \in \{1, 2, \dots, m\}$, let y_i denote the point of the set $(X_i \cup \{x_i\}) \setminus \{x_0\}$, which is the nearest to x_0. Then the simplex $Q_0 = [x_0, y_1, y_2, \dots, y_m]$ can be taken as a starting one for Q.

Applying Lemma and the method of double induction, we get the following statement.

Theorem. *Let a natural number m be greater than or equal to 2 and let Z be a finite set of points in $\mathbf{R}^m$ not contained in an affine hyperplane of $\mathbf{R}^m$. Then Z induces an admissible polyhedron.*

Notice especially that the proof of this theorem is completely constructive and yields a certain geometric algorithm for obtaining the induced admissible polyhedron.

Remark 1. Let P be a simple polygon in the Euclidean plane $\mathbf{R}^2$ and let $v = v(P)$ denote the number of vertices of P. Then there exists a triangulation $\{T_i : i \in I\}$ of P such that the vertices of any triangle T_i are vertices of P. In addition, assuming $v > 3$, there are at least two triangles T_i and T_j such that each of them has two common sides with P. By starting with the latter fact and using induction on v, one can readily demonstrate that any simple polygon in $\mathbf{R}^2$ is admissible. This fact allows also to present an easy proof of the so-called Chvátal's art gallery theorem (see [2], [4]).

Remark 2. In [10], all those finite point systems in $\mathbf{R}^2$ (respectively, in $\mathbf{R}^3$) which determine a unique polygonal curve homeomorphic to $\mathbf{S}_1$ (respectively, a unique polyhedral surface homeomorphic to $\mathbf{S}_2$) were completely described.

Remark 3. Let $\mathbf{N}$ denote the set of all natural numbers and let $f : \mathbf{N} \to \mathbf{N}$ be a non-decreasing function such that

$$f(2n) \leq 2f(n) + an + b$$

for two fixed real numbers $a \geq 0$, $b \geq 0$ and for all $n \in \mathbf{N}$. It can be shown that $f(n) = O(n\log_2(n))$, i.e., there exists a constant $c > 0$ such that $f(n) \leq cn\log_2(n)$ for all natural numbers $n > 1$. For this purpose, we first check (by using induction on $k \in \mathbf{N}$) that

$$f(2^k) \leq 2^k f(1) + k2^k a + (2^k - 1)b.$$

Then, taking any natural number $n > 1$, having found $k \in \mathbf{N}$ such that $2^k \leq n \leq 2^{k+1}$, and applying the above inequality with the monotonicity of f, we get the desired estimation $f(n) = O(n\log_2(n))$. The result just presented is a particular case of the so-called Master Theorem. However, this result efficiently works in many situations and allows to utilize one universal method in various combinatorial constructions. The above-mentioned universal method is usually expressed by the widely known dictatorial phrase: divide and conquer. All approaches based on this method mean that the complexity of a construction of the desired geometric object associated with a given $2n$-point set $Z \subset \mathbf{R}^m$ can be evaluated by the complexities of constructions of the desired geometric objects associated with two suitable n-point sets Z_1 and Z_2 respectively, where $Z_1 \cup Z_2 = Z$ and $Z_1 \cap Z_2 = \emptyset$. For more details, see e.g. [12], [13], [14].

In particular, by using the divide-and-conquer method, it is not difficult to demonstrate that if Z is an arbitrary non-collinear n-element subset of $\mathbf{R}^2$, then a simple polygon induced by Z can be constructed within $O(n\log_2(n))$ steps.

The same method provides us with the following well-known results:

(1) if an abstract n-element set X is given which is linearly ordered by some relation $\preceq$, then $O(n\log_2(n))$ pairs of elements of X suffice to arrange all elements of X according to $\preceq$; in other words, if we know all the induced orderings in $O(n\log_2(n))$ many pairs of elements from X, then we are able to reconstruct the initial ordering $\preceq$ of X;

(2) if Z is a finite subset of $\mathbf{R}^m$ with $\mathrm{card}(Z) = n$, then there exists a geometric algorithm of finding

$$\min\{||z - z'|| : z \in Z, z' \in Z, z \neq z'\}$$

within $O(n\log_2(n))$ steps (notice that here m is fixed and is treated as some constant);

(3) if the dimension m does not exceed 3, and Z is an n-element subset of the space $\mathbf{R}^m$, then a geometric construction of the convex hull of Z is possible for which the number of steps is of order $O(n\log_2(n))$ (see, e.g., [12]).

Remark 4. The result analogous to that of (3) does not longer hold for the space $\mathbf{R}^m$, where $m \geq 4$. Indeed, the complexity of optimal constructions of convex hulls of finite point sets essentially grows, because of the existence in $\mathbf{R}^m$ of the so-called Carathéodory-Gale polyhedra (see [5]).

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