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ON ORDINARY DIFFERENTIAL EQUATIONS WITH BAD RIGHT-HAND SIDES

As is well known, the Cauchy problem (the initial value problem) is formulated as follows: the first order ordinary differential equation

$$x'(t) = f(t, x(t)) \quad ((t_0, x_0) \in \mathrm{dom}(f))$$

is given for some open neighbourhood of the point $(t_0, x_0) \in \mathbf{R}^2$ and a local solution $x(t)$ $(t_0 - \delta \leq t \leq t_0 + \delta)$ from a certain class of functions should be found satisfying the initial condition $x(t_0) = x_0$.

If the function f is continuous in the neighbourhood of (t_0, x_0), then Peano's theorem guarantees the existence of a continuously differentiable local solution (which, in general, is not unique).

Carathéodory's theorem deals with a more general situation where:

(i) $f(t, \cdot)$ is continuous for any $t \in [t_0 - a, t_0 + a]$;

(ii) $f(\cdot, x)$ is Lebesgue measurable for any $x \in [x_0 - b, x_0 + b]$;

(iii) there is a Lebesgue integrable function $m : [t_0 - a, t_0 + a] \to \mathbf{R}$ such that $|f(t, x)| \leq m(t)$ for all $(t, x) \in [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$.

Under these conditions, Caratheodory's theorem states the existence of an absolutely continuous local solution $x(t)$ for which the equality $x'(t) = f(t, x(t))$ holds true for almost all $t \in [t_0 - \delta, t_0 + \delta]$. Obviously, this theorem involves a lot of cases with discontinuous right-hand sides of ordinary differential equations.

However, there are situations where Caratheodory's theorem does not work (see, for instance, [1], [2], [3], [9]). Furthermore, there are many examples of the first order ordinary differential equations with essentially discontinuous right-hand sides, for which the Cauchy problem still makes sense. Briefly speaking, in certain cases, it is possible to consider those first order ordinary differential equations $x'(t) = f(t, x(t))$ for which the function f a priori has a very bad descriptive structure, e.g., f is non-Lebesgue measurable as a function of two variables t and x.

In this direction, for a certain wide class F of functions of two variables, it was demonstrated that if $f \in F$, then for any initial condition $x(t_0) =$

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x_0, there exists a unique locally absolutely continuous solution $x(t)$ of the equation $x'(t) = f(t, x(t))$, satisfying the above initial condition.

Some sup-measurable real-valued functions of two variables can be regarded as representatives from the class F. Notice that, under certain set-theoretical assumptions, the class of sup-measurable functions contains many non-Lebesgue measurable functions (for more details, see [4], [5], [8]).

There are several versions of the concept of sup-measurability. For example, as was shown in [6], the notion of weak sup-measurability is more relevant in studies of questions concerning the existence and uniqueness of local solutions of first order ordinary differential equations with bad right-hand sides.

Here we present a further refinement of the notion of sup-measurability.

Let k be a natural number. As usual, denote by $C^k(\mathbf{R})$ the class of all those real-valued functions on $\mathbf{R}$ which are k-times continuously differentiable.

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function. We shall say that f is (k)-sup-measurable if, for every $h \in C^k(\mathbf{R})$, the superposition

$$t \to f(t, h(t)) \quad (t \in \mathbf{R})$$

is a Lebesgue measurable function of one variable.

Theorem 1. *Assume Martin's Axiom. Then, for any natural number $k > 0$, there exists a Lebesgue measurable (actually, equivalent to zero) (k)-sup-measurable function $f : \mathbf{R}^2 \to \mathbf{R}$ which is not $(k-1)$-sup-measurable.*

In particular, Theorem 1 shows that (under **MA**) there exists a function $f : \mathbf{R}^2 \to \mathbf{R}$ which is sup-measurable with respect to the class of all continuously differentiable functions of one variable, but is not sup-measurable with respect to the class of all continuous functions of one variable (see [6] and [7] for some related results).

Theorem 2. *Suppose Martin's Axiom and let $k > 0$ be a natural number. There exists a function $f : \mathbf{R}^2 \to \mathbf{R}$ satisfying the following relations:*

(1) *f is not Lebesgue measurable;*

(2) *f is (k)-sup-measurable but is not $(k-1)$-sup-measurable;*

(3) *for any initial condition $(t_0, x_0) \in \mathbf{R}^2$, there exists a unique solution of the equation $x'(t) = f(t, x(t))$, in the class of locally absolutely continuous functions, and this solution is a polynomial of the k-th degree such that $x(t_0) = x_0$.*

Theorem 2 shows that there are ordinary differential equations with extremely bad right-hand sides, all the solutions of which belong to the class of polynomials with a fixed nonzero degree.

Notice that the proofs of Theorem 1 and 2 are based on certain properties of generalized Sierpinski subsets of $\mathbf{R}$ and on real-valued continuous nowhere approximately differentiable functions.

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