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TWO–WEIGHT CRITERIA FOR POTENTIALS WITH PRODUCT KERNELS ON CONES OF DECREASING FUNCTIONS

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Our aim in to present two–weight criteria for the following potential operators with product kernels

$$\big(\mathcal{R}_{\alpha_1,\alpha 2}f\big)(x_1,x_2)=\int\limits_0^{x_1}\int\limits_0^{x_2}\frac{f(t_1,t_2)}{(x_1-t_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}}\,dt_1dt_2,$$

$$\big(\mathcal{W}_{\alpha_1,\alpha_2}f\big)(x_1,x_2)=\int\limits_{x_1}^{\infty}\int\limits_{x_2}^{\infty}\frac{f(t_1,t_2)}{(t_1-x_1)^{1-\alpha_1}(t_1-x_1)^{1-\alpha_2}}\,dt_1dt_2,$$

$$(\mathcal{RW})_{\alpha_1,\alpha_2}f(x_1,x_2)=\int\limits_0^{x_1}\int\limits_{x_2}^{\infty}\frac{f(t_1,t_2)dt_1dt_2}{(x_1-t_1)^{1-\alpha_1}(t_2-x_2)^{1-\alpha_2}},$$

$$(\mathcal{WR})_{\alpha_1,\alpha_2}f(x_1,x_2)=\int\limits_{x_1}^{\infty}\int\limits_0^{x_2}\frac{f(t_1,t_2)dt_1dt_2}{(t_1-x_1)^{1-\alpha_1}(x_2-t_2)^{1-\alpha_2}},$$

$$\big(\mathcal{I}_{\alpha_1,\alpha_2)}f\big)(x_1,x_2)=\int\limits_0^{\infty}\int\limits_0^{\infty}\frac{f(t_1,t_2)}{|x_1-t_1|^{1-\alpha_1}|x_2-t_2|^{1-\alpha_2}}\,dt_1dt_2$$

($0<\alpha_1,\alpha_2<1$) on cones of functions f which are non–negative and decreasing in each variable. In our case the right-hand side weight is of product type. The appropriate problem for the one-dimensional potential operator

$$(T_\alpha f)(x)=\int\limits_0^{\infty}\frac{f(t)}{|x-t|^{1-\alpha}}\,dt,\quad 0<\alpha<1,\ x>0,$$

on the cone of decreasing functions is also discussed.

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For the following weighted multiple Riemann–Liouville transform

$$\big(R_{\alpha_1,\dots,\alpha_n} f\big)(x_1,\dots,x_n) = = \frac{1}{\Pi_{i=1}^n x_i^{\alpha_i}} \int\limits_0^{x_1} \cdots \int\limits_0^{x_n} \frac{f(t_1,\dots,t_n)}{\Pi_{i=1}^n (x_i - t_i)^{1-\alpha_i}}\, dt_1 \dots dt_n,$$

we derive one–weight criteria.

We say that a function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is decreasing if f is decreasing in each variable separately. Further, a set $D \subset \mathbb{R}_+^n$ is decreasing if the function χ_D is decreasing.

Let $\mathcal{D}$ be the class of functions $f : \mathbb{R}_+^n \to \mathbb{R}_+$ which are decreasing in each variable separately and let u be measurable a.e. positive function (weight) on $\mathbb{R}_+^n$. We denote by $L^p(u, \mathbb{R}_+^n)$, $0 < p < \infty$, the class of all non–negative functions on $\mathbb{R}_+^n$ for which

$$\|f\|_{L^p(u,\mathbb{R}_+^n)} := \Bigg(\int\limits_{\mathbb{R}_+^n} f^p(x_1,\cdots,x_n) u(x_1,\cdots,x_n) dx_1 \cdots dx_n \Bigg)^{1/p} < \infty.$$

Under the symbol $L^p_{dec}(u, \mathbb{R}_+^n)$ we mean the class $L^p(u, \mathbb{R}_+^n) \cap \mathcal{D}$.

A full characterization of the class of weights u for which the boundedness of the one-dimensional Hardy transform

$$\big(Hf\big)(x) = \frac{1}{x} \int\limits_0^x f(t) dt$$

from $L^p_{dec}(u, \mathbb{R}_+)$ to $L^p(u, \mathbb{R}_+)$ holds, was given in [2]. Two-weight Hardy inequalities on cones of monotonic functions were established in the paper [14]. The multidimensional analogs of these results were studied in [3], [1], [4].

For the weight theory for Hardy–type operators and one-sided potentials we refer e.g., to the monographs [13], [12], [7], [6], [5] and references cited therein. The monograph [11] is dedicated to two–weight criteria for multiple integral operators (see also the papers [8], [9], [10] for criteria guaranteeing trace inequalities for potential operators with multiple kernels).

Together with multiple potential operators we are interested in the one–sided strong fractional maximal operator:

$$\big(\mathcal{M}^-_{\alpha_1,\alpha_2} f\big)(x_1,x_2) = \sup_{\substack{0<h_1\le x_1\\ 0<h_2\le x_2}} h_1^{\alpha_1-1} h_2^{\alpha_2-1} \int\limits_{x_1-h_1}^{x_1} \int\limits_{x_2-h_2}^{x_2} f(t_1,t_2) dt_1 dt_2,$$

where $x_1, x_2 \in \mathbb{R}_+$, $f \ge 0$ and $0 < \alpha_i < 1$, $i = 1, 2$.

Let

$$D_{x_1,\dots,x_n} := D \cap ([0, x_1] \times \cdots \times [0, x_n]), \quad D \subset \mathbb{R}_+^n.$$

The next statement gives one–weight criteria for the operator $R_{\alpha_1,\dots,\alpha_n}$.

Theorem 1. *Let* $0 < p < \infty$ *and let* $0 < \alpha_i < 1$, $i = 1, \dots, n$. *Then* $R_{\alpha_1,\dots,\alpha_n}$ *is bounded from* $L^p_{dec}(u, \mathbb{R}^n_+)$ *to* $L^p(u, \mathbb{R}^n_+)$ *if and only if there is a positive constant* c *such that for all decreasing sets* D, $D \subset \mathbb{R}^n_+$,

$$\int\limits_{\mathbb{R}^n \setminus D} \frac{|D_{x_1,\dots,x_n}|^p}{(x_1 \dots x_n)^p} u(x_1, \dots, x_n) dx_1 \dots dx_n \leq$$

$$\leq c \int\limits_D u(x_1, \dots, x_n) dx_1 \dots dx_n.$$

Let

$$W_j(x_j) := \int\limits_0^{x_j} w_j(t) dt, \quad W(t_1, \dots, t_n) := \Pi_{i=1}^n W_i(t_i);$$

Our results regarding the two-weight problem are given by the following statements.

Theorem 2. *Let* $1 < p \leq q < \infty$ *and let* $0 < \alpha_i < 1$, $i = 1, 2$. *Assume that* v *and* w *are weights on* $\mathbb{R}^2_+$. *Suppose also that* $w(x_1, x_2) = w_1(x_1) w_2(x_2)$ *for some one–dimensional weights* w_1 *and* w_2, *and that* $W_i(\infty) = \infty$, $i = 1, 2$. *Then the following conditions are equivalent:*

(a) $\mathcal{R}_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w, \mathbb{R}^2_+)$ *to* $L^q(v, \mathbb{R}^2_+)$;

(b) $\mathcal{M}^-_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w, \mathbb{R}^2_+)$ *to* $L^q(v, \mathbb{R}^2_+)$;

(c) *the following four conditions hold simultaneously:*

$$\sup_{a_1,a_2>0} \left(\int\limits_0^{a_1} \int\limits_0^{a_2} w(t_1, t_2) dt_1 dt_2 \right)^{-1/p} \times$$

$$\times \left(\int\limits_0^{a_1} \int\limits_0^{a_2} \left(t_1^{\alpha_1} t_2^{\alpha_2} \right)^q v(t_1, t_2) dt_1 dt_2 \right)^{1/q} < \infty; \tag{1}$$

$$\sup_{a_1,a_2>0} \left(\int\limits_0^{a_1} \int\limits_0^{a_2} (t_1 t_2)^{p'} W^{-p'}(t_1, t_2) w(t_1, t_2) dt_1, dt_2 \right)^{1/p'} \times$$

$$\times \left(\int\limits_{a_1}^{\infty} \int\limits_{a_2}^{\infty} \left(t_1^{\alpha_1 - 1} t_2^{\alpha_2 - 1} \right)^q v(t_1, t_2) dt_1 dt_2 \right)^{1/q} < \infty; \tag{2}$$

$$\sup_{a_1,a_2>0} \left(\int\limits_0^{a_1} w_1(t_1) dt_1 \right)^{-1/p} \left(\int\limits_0^{a_2} t_2^{p'} W_2^{-p'}(t_2) w_2(t_2) dt_2 \right)^{1/p'} \times$$

$$\times\left(\int_0^{a_1}\int_{a_2}^{\infty} t_1^{q\alpha_1} t_2^{q(\alpha_2-1)} v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty; \tag{3}$$

$$\sup_{a_1,a_2>0}\left(\int_0^{a_1} t_1^{p'} W_1^{-p'}(t_1)w_1(t_1)dt_1\right)^{1/p'}\left(\int_0^{a_2} w_2(t_2)dt_2\right)^{-1/p}\times$$

$$\times\left(\int_{a_1}^{\infty}\int_0^{a_2} t_1^{q(\alpha_1-1)} t_2^{q\alpha_2} v(t_1,t_2)dt_1dt_2\right)^{1/q} < \infty. \tag{4}$$

Analogous result for the double Hardy operator H_2 was derived in [3] in the case when both v and w are product weights.

Corollary 1. *Let $1 < p \le q < \infty$ and let $0 < \alpha_i < 1$, $i = 1, 2$. Then the following conditions are equivalent:*

(a) *the boundedness of $\mathcal{R}_{\alpha_1,\alpha_2}$ from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ holds for $w \equiv 1$;*

(b) *the operator $\mathcal{M}^-_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ for $w \equiv 1$;*

(c)

$$\sup_{a_1,a_2>0}(a_1a_2)^{1/p'}\left(\int_{a_1}^{\infty}\int_{a_2}^{\infty} x_1^{q(\alpha_1-1)} x_2^{q(\alpha_2-1)} v(x_1,x_2)dx_1dx_2\right)^{1/q} < \infty.$$

Theorem 3. *Let $1 < q < p < \infty$ and let $0 < \alpha_i < 1$, $i = 1, 2$. Assume that v and w are weights on $\mathbb{R}^2_+$. Suppose also that $w(x_1,x_2) = w_1(x_1)w_2(x_2)$ and that $W_i(\infty) = \infty$, $i = 1, 2$. Then the following conditions are equivalent:*

(a) *$\mathcal{R}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$;*

(b) *$\mathcal{M}^-_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$;*

(c) *the following four conditions hold:*

$$\left[\int_{\mathbb{R}^2_+}\left(\int_0^{t_1}\int_0^{t_2} v(x_1,x_2)\left(x_1^{\alpha_1}x_2^{\alpha_2}\right)^q dx_1dx_2\right)^{r/q}\times\right.$$

$$\left.\times W^{-r/q}(t_1,t_2)w(t_1,t_2)dt_1dt_2\right]^{1/r} < \infty;$$

$$\left[\int_{\mathbb{R}^2_+}\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} v(x_1,x_2)\left(x_1^{\alpha_1-1}x_2^{\alpha_2-1}\right)^q dx_1dx_2\right)^{r/q}\times\right.$$

$$\times\left(\int_0^{t_1}\int_0^{t_2}(x_1x_2)^{p'}W^{-p'}(x_1,x_2)w(x_1,x_2)dx_1dx_2\right)^{r/q'}\times$$

$$\times(t_1t_2)^{p'}W^{-p'}(t_1,t_2)w(t_1,t_2)dt_1dt_2\Bigg]^{1/r}<\infty;$$

$$\Bigg[\int\limits_{\mathbb{R}^2_+}\Bigg(\int\limits_0^{t_1}\int\limits_{t_2}^{\infty}v(x_1,x_2)\Big(x_1^{\alpha_1}x_2^{\alpha_2-1}\Big)^q dx_1dx_2\Bigg)^{r/q}W_1^{-r/q}(t_1)\times$$

$$\times\Bigg(\int\limits_0^{t_2}x_2^{p'}W_2^{-p'}(x_2)w_2(x_2)dx_2\Bigg)^{r/q'}t_2^{p'}W_2(t_2)w_2(t_2)dt_1dt_2\Bigg]^{1/r}<\infty;$$

$$\Bigg[\int\limits_{\mathbb{R}^2_+}\Bigg(\int\limits_{t_1}^{\infty}\int\limits_0^{t_2}v(x_1,x_2)\Big(x_1^{\alpha_1-1}x_2^{\alpha_2}\Big)^q dx_1dx_2\Bigg)^{r/q}W_2^{-r/q}(t_2)\times$$

$$\times\Bigg(\int\limits_0^{t_1}x_1^{p'}W_1^{-p'}(x_1)w_1(x_1)dx_1\Bigg)^{r/q'}t_1^{p'}W_1(t_1)w_1(t_1)dt_1dt_2\Bigg]^{1/r}<\infty,$$

where $1/r=1/q-1/p$.

Theorem 4. *Let* $1<p\le q<\infty$ *and let* $0<\alpha_1,\alpha_2\le 1$. *Suppose that the weight function* w *on* $\mathbb{R}^2_+$ *is of product type, i.e.* $w(x_1,x_2)=w_1(x_1)w_2(x_2)$. *Suppose also that* $W_1(\infty)=W_2(\infty)=\infty$.

(i) *The operator* $(\mathcal{RW})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w,\mathbb{R}^2_+)$ *to* $L^q(v,\mathbb{R}^2_+)$ *if and only if*

$$\sup_{a,b>0}\Bigg(\int\limits_0^a\int\limits_0^b\frac{x_1^{\alpha_1 q}v(x_1,x_2)}{(b-x_2)^{-\alpha_2 q}}\,dx_1dx_2\Bigg)^{1/q}\times$$

$$\times\Bigg(\int\limits_0^a\int\limits_0^b w_1(x_1)w_2(x_2)dx_1dx_2\Bigg)^{-1/p}<\infty;\qquad(5)$$

$$\sup_{a,b>0}\Bigg(\int\limits_0^a\int\limits_0^b x_1^{\alpha_1 q}v(x_1,x_2)dx_1dx_2\Bigg)^{1/q}\times$$

$$\times\Bigg(\int\limits_0^a w_1(x_1)dx_1\Bigg)^{-1/p}\Bigg(\int\limits_b^{\infty}W_2^{-p'}(x_2)w_2(x_2)(x_2-b)^{\alpha_2 p'}dx_2\Bigg)^{1/p'}<\infty;\quad(6)$$

$$\sup_{a,b>0}\Bigg(\int\limits_a^{\infty}\int\limits_0^b\frac{v(x_1,x_2)}{x_1^{(1-\alpha_1)q}(b-x_2)^{-\alpha_2 q}}\,dx_1dx_2\Bigg)^{1/q}\times$$

$$\times\Bigg(\int\limits_0^a x_1^{p'}W_1^{-p'}(x_1)w_1(x_1)dx_1\Bigg)^{1/p'}\Bigg(\int\limits_0^b w_2(x_2)dx_2\Bigg)^{-1/p}<\infty;\qquad(7)$$

$$\sup_{a,b>0}\Big(\int_a^\infty\int_0^b x_1^{(\alpha_1-1)q}v(x_1,x_2)dx_1dx_2\Big)^{1/q}\times$$

$$\times\Big(\int_0^a\int_b^\infty \frac{W^{-p'}(x_1,x_2)w(x_1,x_2)x_1^{p'}}{(x_2-b)^{-\alpha_2 p'}}\,dx_1dx_2\Big)^{1/p'}<\infty. \tag{8}$$

(ii) *The operator* $(\mathcal{WR})_{\alpha_1,\alpha_2}$ *is bounded from* $L^p_{dec}(w,\mathbb{R}^2_+)$ *to* $L^q(v,\mathbb{R}^2_+)$ *if and only if*

$$\sup_{a,b>0}\Big(\int_0^a\int_0^b \frac{x_2^{\alpha_2 q}v(x_1,x_2)}{(a-x_1)^{-\alpha_1 q}}\,dx_1dx_2\Big)^{1/q}\times$$

$$\times\Big(\int_0^a\int_0^b w_1(x_1)w_2(x_2)dx_1dx_2\Big)^{-1/p}; \tag{9}$$

$$\sup_{a,b>0}\Big(\int_0^a\int_0^b x_2^{\alpha_2 q}v(x_1,x_2)dx_1dx_2\Big)^{1/q}\Big(\int_0^b w_2(x_2)dx_2\Big)^{-1/p}\times$$

$$\times\Big(\int_a^\infty W_1^{-p'}(x_1)w_1(x_1)(x_1-a)^{\alpha_1 p'}dx_1\Big)^{1/p'}<\infty; \tag{10}$$

$$\sup_{a,b>0}\Big(\int_0^a\int_b^\infty \frac{v(x_1,x_2)}{x_2^{(1-\alpha_2)q}(a-x_1)^{-\alpha_1 q}}\,dx_1dx_2\Big)^{1/q}\times$$

$$\times\Big(\int_0^a w_1(x_1)dx_1\Big)^{-1/p}\Big(\int_0^b x_2^{p'}W_2^{-p'}(x_2)w_2(x_2)dx_2\Big)^{1/p'}<\infty; \tag{11}$$

$$\sup_{a,b>0}\Big(\int_0^a\int_b^\infty x_2^{(\alpha_2-1)q}v(x_1,x_2)dx_1dx_2\Big)^{1/q}\times$$

$$\times\Big(\int_a^\infty\int_0^b \frac{W^{-p'}(x_1,x_2)w(x_1,x_2)x_2^{p'}}{(x_1-a)^{-\alpha_1 p'}}\,dx_1dx_2\Big)^{1/p'}<\infty. \tag{12}$$

Definition 1. We say that a locally integrable a.e. positive function ρ on $\mathbb{R}^2$ satisfies the doubling condition with respect to the second variable ($\rho\in DC(y)$) if there is a positive constant c such that for all $t>0$ and almost every $x>0$ the following inequality holds:

$$\int_0^{2t}\rho(x,y)dy\le c\min\Big\{\int_0^t\rho(x,y)dy,\int_t^{2t}\rho(x,y)dy\Big\}.$$

Analogously is defined the class of weights $DC(x)$.

Theorem 5. *Let $1<p\le q<\infty$ and let $0<\alpha_1,\alpha_2\le 1$. Suppose that the weight function w on $\mathbb{R}^2_+$ is of product type, i.e. $w(x_1,x_2)=w_1(x_1)w_2(x_2)$. Suppose also that $W_1(\infty)=W_2(\infty)=\infty$.*

(i) *If $v\in DC(y)$, then $\mathcal{W}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if*

$$\sup_{a,b>0}\Bigg(\int_0^a\int_0^b v(x_1,x_2)(a-x_1)^{\alpha_1 q}dx_1dx_2\Bigg)^{1/q}\times$$

$$\times\Bigg(\int_0^a w_1(x_1)dx_1\Bigg)^{-1/p}\Bigg(\int_b^\infty W_2^{-p'}(x_2)w_2(x_2)x_2^{\alpha_2 p'}dx_2\Bigg)^{1/p'}<\infty; \qquad (13)$$

$$\sup_{a,b>0}\Bigg(\int_0^a\int_0^b v(x_1,x_2)dx_1dx_2\Bigg)^{1/q}\times$$

$$\times\Bigg(\int_a^\infty\int_b^\infty W^{-p'}(x_1,x_2)w(x_1,x_2)(x_1-a)^{\alpha_1 p'}x_2^{\alpha_2 p'}dx_1dx_2\Bigg)^{1/p'}<\infty; \qquad (14)$$

(ii) *If $v\in DC(x)$, then $\mathcal{W}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if*

$$\sup_{a,b>0}\Bigg(\int_0^a\int_0^b v(x_1,x_2)(b-x_2)^{\alpha_2 q}dx_1dx_2\Bigg)^{1/q}\times$$

$$\times\Bigg(\int_a^\infty W_1^{-p'}(x_1)w_1(x_1)x_1^{\alpha_1 p'}dx_1\Bigg)^{1/p'}\Bigg(\int_0^b w_2(x_2)dx_2\Bigg)^{-1/p}<\infty; \qquad (15)$$

$$\sup_{a,b>0}\Bigg(\int_0^a\int_0^b v(x_1,x_2)dx_1dx_2\Bigg)^{1/q}\times$$

$$\times\Bigg(\int_a^\infty\int_b^\infty W^{-p'}(x_1,x_2)w(x_1,x_2)(x_2-b)^{\alpha_2 p'}x_1^{\alpha_1 p'}dx_1dx_2\Bigg)^{1/p'}<\infty. \qquad (16)$$

Theorem 6. *Let $1<p\le q<\infty$ and let $0<\alpha_1,\alpha_2<1$. Suppose that the weight v belongs to the class $DC(y)$. Let $w(x_1,x_2)=w_1(x_1)w_2(x_2)$ for some one-dimensional weight functions w_1 and w_2 and $W_1(\infty)=W_2(\infty)=\infty$. Then the operator $\mathcal{I}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w,\mathbb{R}^2_+)$ to $L^q(v,\mathbb{R}^2_+)$ if and only if conditions $(1)-(14)$ are satisfied.*

Theorem 7. *Let $1 < p \le q < \infty$ and let $0 < \alpha_1, \alpha_2 < 1$. Suppose that the weight v belongs to the class $DC(x)$. Let $w(x_1, x_2) = w_1(x_1)w_2(x_2)$ for some one-dimensional weight functions w_1 and w_2 and $W_1(\infty) = W_2(\infty) = \infty$. Then the operator $\mathcal{I}_{\alpha_1,\alpha_2}$ is bounded from $L^p_{dec}(w, \mathbb{R}^2_+)$ to $L^q(v, \mathbb{R}^2_+)$ if and only if conditions (1) – (12), (15) and (16) are satisfied.*

Finally we discuss the two–weight problem for one-dimensional potential:

$$T_\alpha f(x) = \int\limits_0^\infty \frac{f(t)}{|x-t|^{1-\alpha}}\, dt, \quad 0 < \alpha < 1,\ x > 0,$$

on the cone of one–dimensional decreasing functions.

We denote $W(x) := \int_0^x w(t)dt$.

Theorem 8. *Let $1 < p \le q < \infty$ and let $0 < \alpha < 1$. Then T_α is bounded from $L^p_{dec}(w, \mathbb{R})$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$\sup_{a>0} \Big(\int\limits_0^a w(t)dt \Big)^{-1/p} \Big(\int\limits_0^a t^{\alpha q} v(t)dt \Big)^{1/q} < \infty;$$

$$\sup_{a>0} \Big(\int\limits_0^a t^{p'} W^{-p'}(t) w(t)dt \Big)^{1/p'} \Big(\int\limits_a^\infty t^{(\alpha-1)q} v(t)dt \Big)^{1/q} < \infty;$$

$$\sup_{a>0} \Big(\int\limits_a^\infty W^{-p'}(x) w(x) (x-a)^{\alpha p'}\, dx \Big)^{1/p'} \Big(\int\limits_0^a v(x)dx \Big)^{1/q} < \infty;$$

$$\sup_{a>0} \Big(\int\limits_0^a w(x)dx \Big)^{-1/p} \Big(\int\limits_0^a v(x)(x-a)^{\alpha q} dx \Big)^{1/q} < \infty.$$

Theorem 9. *Let $1 < q < p < \infty$ and let $0 < \alpha < 1$. Then T_α is bounded from $L^p_{dec}(w, \mathbb{R})$ to $L^q(v, \mathbb{R}_+)$ if and only if*

$$\Big[\int\limits_{\mathbb{R}_+} \Big[\Big(\int\limits_0^t x^{\alpha q} v(x)dx \Big)^{1/p} W^{-1/p}(t) \Big]^r v(t)dt \Big]^{1/r} < \infty;$$

$$\Big[\int\limits_{\mathbb{R}_+} \Big[\Big(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}} dx \Big)^{1/p} \Big(\int\limits_0^t \frac{W^{-p'}(x) w(x)}{x^{-p'}} \Big)^{1/p'} \Big]^r \times$$

$$\times t^{p'} W^{-p'}(t) w(t) dt \Big]^{1/r} < \infty;$$

$$\Big[\int\limits_{\mathbb{R}_+}\Big[\Big(\int\limits_t^\infty \frac{W^{-p'}(x)w(x)}{(x-t)^{-\alpha p'}}\Big)^{1/p'}\Big(\int\limits_0^t v(x)dx\Big)^{1/p}\Big]^r v(t)dt\Big]^{1/r}<\infty;$$

$$\Big[\int\limits_{\mathbb{R}_+}\Big(\int\limits_t^\infty W^{-1/p}(t)\Big(\int\limits_0^t \frac{v(x)}{(t-x)^{-\alpha q}}dx\Big)^{1/q}\Big]^r W^{-p'}(t)w(t)dt\Big]^{1/r}<\infty,$$

where $1/r = 1/q - 1/p$.

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