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THE HASEMAN BOUNDARY VALUE PROBLEM IN CLASSES OF FUNCTIONS REPRESENTABLE BY THE CAUCHY TYPE INTEGRAL WITH DENSITY IN $L^{p(\cdot)}$

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Let Γ be a simple closed rectifiable oriented curve dividing a complex plane into the domains D^+ with the origin, and D^- with a point at infinity. The Haseman boundary value problem is called that which is formulated as follows:

Find an analytic on the plane divided along Γ function Φ from the given class of functions A possessing the boundary values $\Phi^+(t)$ in D^+ and $\Phi^-(t)$ in D^-, $t \in \Gamma$ and satisfying the condition

$$\Phi^+(\alpha(t)) = a(t)\,\Phi^-(t) + b(t), \tag{1}$$

where $a(t)$, $b(t)$ are the given on Γ functions, and $\alpha(t)$ is the orientation–preserving homeomorphism of Γ onto itself.

C. Haseman for the first time investigated the homogeneous problem

$$\Phi^+(\alpha(t)) = a(t)\,\Phi^-(t) \tag{2}$$

[1]. Later on, the problem with a shift was considered by T. Carleman [2].

A complete solution of the problem (1) under the assumptions that Γ is the Ljapunov curve, $\alpha'(t)$, $a(t)$, $b(t)$ are the functions from Hölder class, $\alpha'(t)$ and $a(t)$ are other than zero, and Φ is piecewise holomorphic, has been given by D. A. Kveselava ([3], [4]).

N. P. Vekua considered the problem (1) in a vector case. His results as well as those of the other authors in this direction can be found in [5].

Subsequently, the problems with a shift became a subject of investigation by various authors. A great deal of such kind of problems are reflected in the book due to G. S. Litvinchuk [6].

The case in which in the capacity of the class A appears the $K^p(\Gamma)$–class of functions representable by the Cauchy type integral with density from the Lebesgue class $L^p(\Gamma)$, $p > 1$ was considered by B. V. Khvedelidze [7],

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B. V. Khvedelidze and G. F. Manjavidze [8], I. B. Simonenko [9], and G. F. Manjavidze [10].

The methods of investigation of the problem (1) expounded in the book of G.F. Manjavidze [10], thanks to the results obtained for singular and Cauchy type integrals with densities from Lebesgue spaces with a variable exponent ([11]–[13]) and also due to Lemmas 1–4 cited below, make it possible to extend the well-known results concerning Haseman's problem to more general cases.

The results we present in this paper are analogous to those obtained for the case in which Γ is the Ljapunov curve and $A = K^p(\Gamma)$, however, for the first time we have covered the case of curves with a chord-arc-condition and also the case in which $A = K^{p(\cdot)}(\Gamma)$–class of Cauchy type integrals with densities from the Lebesgue space $L^{p(\cdot)}(\Gamma)$ with a variable exponent $p(t)$ take the part of unknown functions.

1^0. Curves with the Chord-arc-Condition and One of Their Properties

Let Γ be a simple rectifiable curve and $z = z(\zeta)$, $0 \le \zeta \le l$, be its equation with respect to the arc abscissa. We say that Γ is a curve with the chord-arc-condition and write $\Gamma \in HC$, if there exists a constant $m > 0$ such that for any $t, \tau \in \Gamma$

$$|t-\tau| \ge m\, s(t,\tau), \tag{3}$$

where $s(t,\tau)$ is the length of the least of two arcs lying on Γ and connecting the points t and τ.

Further, let $\alpha = \alpha(t)$ be an orientation-preserving homeomorphism of Γ onto itself for which at every point of Γ there exists the derivative $\alpha'(t) \ne 0$, and $\alpha'(t) \in H(\mu)$, i.e., there exist the constants M, μ, $0 < \mu \le 1$ such that

$$\big|\alpha'(t_1) - \alpha'(t_2)\big| < M|t_1 - t_2|^{\mu}, \quad t_1,\, t_2 \in \Gamma. \tag{4}$$

Lemma 1. *Let $\Gamma \in HC$ and $\alpha = \alpha(t)$ be the orientation–preserving homeomorphism of Γ onto itself, $\alpha'(t) \ne 0$, the condition* (4) *be fulfilled and*

$$K(\tau,t) = \frac{\alpha'(t)}{\alpha(\tau)-\alpha(t)} - \frac{1}{\tau - t}\,, \quad t,\, \tau \in \Gamma.$$

Then there exists the constant C such that

$$\big|K(\tau,t)\big| < C\big[s(\tau,t)\big]^{\mu-1}\,. \tag{5}$$

2^0. The Classes of Functions $\mathcal{P}$, $\widetilde{K}_{\alpha}^{p(\cdot)}(\Gamma)$ and $K_{\alpha}^{p(\cdot)}(\Gamma)$

Let $p : \Gamma \to R^+$ be the real function satisfying the following conditions:

1) there exists the constant A such that for every $t_1, t_2 \in \Gamma$ we have

$$\big|p(t_1) - p(t_2)\big| < \frac{A}{\big|\ln|t_1 - t_2|\big|}\,;$$

2) $p_0 = \min\limits_{t\in\Gamma} p(t) > 1$.

A set of all such functions we denote by $\mathcal{P}$.

By $L^{p(\cdot)}(\Gamma)$ we denote a set of measurable on Γ functions f for which

$$I_{p(\cdot)}(f) = \int\limits_{\Gamma} \big|f(t)\big|^{p(t)} |dt| = \int\limits_{0}^{l} \big|f(t(\zeta))\big|^{p(t(\zeta))} d\zeta < \infty.$$

If $p \in \mathcal{P}$, then $L^{p(\cdot)}(\Gamma)$ is the Banach space with the norm

$$\|f\|_{p(\cdot)} = \inf\Big\{\lambda > 0 :\ I_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \le 1\Big\}.$$

A set of the functions $\Phi(z)$, analytic in the plane divided along Γ and representable in the form

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(t)\,dt}{t-z} + q(z), \quad z\overline{\in}\Gamma, \quad f \in L^{p(\cdot)}(\Gamma), \quad p \in \mathcal{P}, \tag{6}$$

where $q(z)$ is a polynomial, we denote by $\widetilde{K}^{p(\cdot)}(\Gamma)$.

A subset of the functions from $\widetilde{K}^{p(\cdot)}(\Gamma)$ for which $q(z) = 0$, is denoted by $K^{p(\cdot)}(\Gamma)$.

Let $p_\alpha(t) = \max(p(t), p(\alpha(t))$ and let

$$\widetilde{K}^{p(\cdot)}_{\alpha}(\Gamma) = \Big\{\Phi \in \widetilde{K}^{p(\cdot)}(\Gamma) : \Phi^- \in L^{p_\alpha(\cdot)}(\Gamma)\Big\},$$
$$K^{p(\cdot)}_{\alpha}(\Gamma) = \Big\{\Phi \in K^{p(\cdot)}(\Gamma) : \Phi^- \in L^{p_\alpha(\cdot)}(\Gamma)\Big\}.$$

If $p(t) \in \mathcal{P}$ then function $p_\alpha(t) \in \mathcal{P}$.

If Γ is the Carleson curve (in particxular, if it satisfies the chord-arc-condition), and $p(z) \in \mathcal{P}$, then the function $\Phi \in \widetilde{K}^{p(\cdot)}(\Gamma)$ has almost for all $t \in \Gamma$ angular boundary values $\Phi^+(t)$ and $\Phi^-(t)$, and the functions $\Phi^\pm$ belong to $L^{p(\cdot)}(\Gamma)$ ([12]). Moreover, since $K^{p(\cdot)}(\Gamma) \subset K^{p_0}(\Gamma)$, therefore $\Phi(z) \in E^{p_0}(D^+)$, while $\big[\Phi(z) - q(z)\big] \in E^{p_0}(D^-)$ (see, for e.g., [14], p. 29), ($E^p(D)$ is the Smirnov class of analytic in D functions).

3^0. Statement of the Problem

We consider the problem (1) in the following statement.

Let $\Gamma \in HC$, $\alpha = \alpha(t)$ be the orientation-preserving homeomorphism of Γ onto itself, $\alpha'(t) \neq 0$; $\alpha'(t) \in H(\mu)$; $a(t)$ be a continuous, other than zero function; $p(t) \in \mathcal{P}$ and $b(t) \in L^{p_\alpha(\cdot)}(\Gamma)$. We are required to find the function $\Phi \in K_\alpha^{p(\cdot)}(\Gamma)$ whose angular boundary values Φ^+ and Φ^- satisfy almost everywhere on Γ the boundary condition (1).

Assume

$$\varkappa = \operatorname{ind} a(t) = \frac{1}{2\pi}\big[\arg a(t)\big]_\Gamma, \tag{7}$$

where $[\ ]_\Gamma$ is an increment of the function appearing in the square brackets for a single circuit around the curve Γ by the point t.

Below, with the rare exception, we will not specify the assumptions regarding the given and unknown elements of the problem under consideration.

4^0. The Conditions of Solvability of the Problem (1)

On the basis of the properties characteristic of the functions from $K_\alpha^{p(\cdot)}(\Gamma)$, it is not difficult to show that any solution Φ of the problem (1) is representable in the form

$$\Phi(z) = \begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{\mu(t)\,dt}{t-z}, & z \in D^-, \\ \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{a(\beta(t))\mu(\beta(t)) + b(\beta(t))}{t-z}\,dt, & z \in D^+, \end{cases} \tag{8}$$

where $\mu \in L^{p_\alpha(\cdot)}(\Gamma)$, and $\beta(t)$ is the inverse to $\alpha(t)$ function.

Taking into account the boundary conditions, we obtain for μ the integral equation in the class $L^{p_\alpha(\cdot)}(\Gamma)$:

$$K(a)\mu \equiv \frac{2a(t_0)}{\pi i}\int_\Gamma \frac{\mu(t)\,dt}{t-t_0} + T(a)\mu + M(a)\mu = \widetilde{b}(t_0), \ \ t_0 \in \Gamma, \tag{9}$$

where

$$T(a)\mu = \frac{1}{\pi i}\int_\Gamma \frac{a(t)-a(t_0)}{t-t_0}\,\mu(t)\,dt, \quad M(a)\mu = \frac{1}{\pi i}\int_\Gamma K(t_0,t)\,a(t)\,\mu(t)\,dt,$$

$$\widetilde{b}(t_0) = b(t_0) - \frac{1}{\pi i}\int_\Gamma \frac{b(\tau)\,d\tau}{t-t_0} - \frac{1}{\pi i}\int_\Gamma K(t_0,\tau)\,b(\tau)\,d\tau.$$

Every solution of equation (9) generates by means of formula (8) a solution of the problem (1).

Just as for the constant p, following [15] (p. 85), we state that the operator $T(a)\mu$ is completely continuous in $L^{p(\cdot)}(\Gamma)$. By virtue of Lemma 1, using the [13], we obtain a completely continuity of the operator $M(a)\mu$, as well. Therefore the operator $K(a)\mu$ is the Noetherian one in $L^{p_\alpha(\cdot)}(\Gamma)$

([11]). Consequently, equation (9) is solvable only for those $\widetilde{b}(t)$ for which

$$\int_\Gamma \widetilde{b}(t)\, V(t)\, dt = 0, \tag{10}$$

where $V(t)$ is an arbitrary solution of the class $L^{p'_\alpha(\cdot)}(\Gamma)$, $p'_\alpha(t) = \frac{p_\alpha(t)}{p_\alpha(t)-1}$ of the equation

$$K'(a)V = -\frac{1}{\pi i}\int_\Gamma \frac{a(t)+a(t_0)}{t-t_0}\, V(t)\, dt - \frac{a(t_0)}{\pi i}\int_\Gamma K(t,t_0)\, V(t)\, dt = 0. \tag{11}$$

Thus the following lemma is valid.

Lemma 2. *The problem* (1) *is solvable if and only if the condition* (10)*, where $V(t)$ is an arbitrary solution of the class $L^{p'_\alpha(\cdot)}(\Gamma)$ of equation* (11)*, is fulfilled.*

5^0. Some Properties of the Solution of the Problem $\omega^+(\alpha(t)) = \omega^-(t)$.

Let Γ, be the simple closed rectifiable curve bounding the domains D^+ and D^-. If $\omega^+(z)$ is holomorphic in D^+ and continuous in $\overline{D^+}$, $\omega^-(z) = Az + \omega_0^-(z)$, where $A = \text{const} \neq 0$, $\omega_0^-(z)$ is holomorphic in D^- and continuous in $\overline{D^-}$ and everywhere on Γ the condition $\omega^+(\alpha(t)) = \omega^-(t)$ is fulfilled, then it was proved in [10], (pp. 77–79) that the functions $\omega^+(z)$ and $\omega^-(z)$ are schlicht in the domains $\overline{D^+}$ and $\overline{D^-}$, respectively, and the curve $\gamma = \omega^-(\Gamma)$ is simple. If, however, $A = 0$, then $\omega^+(z) = C$, $\omega^-(z) = C$, where C is the constant.

In addition to the above-said, we state that the following lemma is valid.

Lemma 3. *If $\Gamma \in HC$, then $\big[\omega^+(z)\big]' \in \underset{\delta>1}{E}{}^{\delta}(D^+)$ and $\big(\big[\omega^-(z)\big]' - A\big) \in \bigcap_{\delta>1} E^\delta(D^-)$, and the curve γ is rectifiable one.*

6^0. Solution of the Homogeneous Problem. Canonical Solution

Lemma 4. *All solutions of the problem* (2) *of the class $\widetilde{K}^\lambda(\Gamma)$, $\lambda > 1$ belong to the class $\bigcap_{\delta>1} \widetilde{K}^\delta(\Gamma)$.*

To prove the lemma, we first, analogously to [9] and [10], state that for the coefficient $a(t)$, close to unity, the problem (2) has a solution $X_0(z)$ possessing the following properties:

$$X_0(z),\ \ \big[X_0(z)\big]^{-1} \in \bigcap_{\delta>1} \widetilde{K}^\delta,\ \ X_0(\infty) = 1. \tag{12}$$

Next, approaching $a(t)$ by means of the rational functions of the type $r(t) = \sum\limits_{k=-n}^{m} (t - z_0)^k$, $z_0 \in D^+$, the condition (2) can be written in the form

$$\Phi^+(\alpha(t)) = a_0(t)\, r(t)\, \Phi^-(t), \quad a_0(t) = a(t)\, r^{-1}(t),$$

and using essentially the results of Lemma 3, we establish that all solutions of the problem (2) of the class $\widetilde{K}^\lambda(\Gamma)$ are given by the equality

$$\Phi(z) = \begin{cases} \big[X_0^+(z)\, \widetilde{q}(\omega^+(z))\big] P(\omega^+(z)), & z \in D^+, \\ \big[X_0^-(z)\, r^{-1}(z)\, \widetilde{q}(\omega^-(z))\big] P(\omega^-(z)), & z \in D^-, \end{cases} \tag{13}$$

where $\widetilde{q}(z)$ is the definite polynomial such that $r^{-1}(z)\, \widetilde{q}(\omega^-(z))$ has no poles in D^-, and $P(z)$ is an arbitrary polynomial. Every function given by equation (13) belongs to the class $\bigcap\limits_{\delta>1} \widetilde{K}^\delta(\Gamma)$.

The function

$$X(z) = \begin{cases} X_0^+(z)\, \widetilde{q}(\omega^+(z)), & z \in D^+, \\ X_0^-(z)\, r^{-1}(z)\, \widetilde{q}(\omega^-(z)), & z \in D^-, \end{cases} \tag{14}$$

is called a canonical solution of the problem (2). The functions $X(z)$ and $[X(z)]^{-1}$ belong to $\bigcap\limits_{\delta>1} \widetilde{K}^\delta(\Gamma)$. It can be shown that $X(z)$ is of order $(-\varkappa)$ at infinity. All the solutions with such properties differ from each other only by a constant multiplier.

7^0. Solution of the Problem (1) in the Classes $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma)$ and $K_\alpha^{p(\cdot)}(\Gamma)$

Theorem. *All the solutions of the class $\widetilde{K}^p(\Gamma)$ of the problem (1) are given by the formula*

$$\Phi(z) = X(z)\big[\Phi_0(z) + P(\omega(z))\big], \tag{15}$$

where $X(z)$ is the canonical solution of the problem (2) (given by equality (14)), and $\Phi_0(z)$ is the solution of the problem

$$\Phi_0^+(\alpha(t)) = \Phi_0^-(t) + b_0(t), \quad b_0(t) = \big[X_0^+(\alpha(t))\big]^{-1} b(t), \tag{16}$$

belonging to the class $\widetilde{K}_\alpha^{p(\cdot)-\varepsilon}(\Gamma)$, and $P(z)$ is an arbitrary polynomial.

Solutions vanishing at infinity, i.e., those of the class $K_\alpha^{p(\cdot)}(\Gamma)$, always exist for $\varkappa \geq 0$, and a general solution is given by equality (15) in which $P(z) = P_{\varkappa-1}(z)$ is an arbitrary polynomial of order $\varkappa - 1$, ($P_{-1}(z) \equiv 0$). If, however, $\varkappa < 0$, then for the problem to be solvable, it is necessary and sufficient that the condition

$$\int\limits_\Gamma t^k \mu(t)\, dt = 0, \quad k = 0, 1, \ldots, |\varkappa| - 1, \tag{17}$$

be fulfilled. Here $\mu(t)$ *is a solution of the equation* $K(1)\mu = b(t)\big[X_0^+(\alpha(t))\big]^{-1}$. *If the conditions* (17) *are fulfilled, the problem has a unique solution given by equality* (15) *with* $P(t) \equiv 0$.

In proving the above theorem, the problem (1) in the class $\widetilde{K}^p_\alpha(\Gamma,)$ along with the canonical solution reduces first to the problem (16), and we construct the solution $\Psi(z) = \Phi_0(z)\, X(z)$ of the class $\widetilde{K}^{p(\cdot)-\varepsilon}(\Gamma)$. Then we establish that the problem is solvable in $\widetilde{K}^{p(\cdot)}(\Gamma)$ for arbitrary $b(t) \in L^{p_\alpha(\cdot)}(\Gamma)$. The difference of that solution and $\Psi(z)$ is a solution of the problem (2). By virtue of Lemma 4 we can conclude that $\Psi \in \widetilde{K}^p_\alpha(\Gamma)$, and using formula (13), we construct all solutions of the problem (1) of the class $\widetilde{K}^{p(\cdot)}_\alpha(\Gamma)$.

In the future report I present the results on Haseman's problem with discontinuous coefficients.

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