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ON THE SUMMABILITY METHODS DEFINED BY MATRIX OF FUNCTIONS

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Let for every nonegative integer numbers n and k a measurable function $a_{n,k}(t)$ is defined on $E = [a, b]$.

We consider a matrix constructed by $a_{n,k}(t)$ functions, i. e.

$$M(t) = \|a_{n,k}(t)\| \text{ where } t \in E.$$

Let these functions satisfy the Toeplitz conditions in any point $t \in E$, i. e.

i) $\lim\limits_{n\to\infty} a_{n,k}(t) = 0, \quad t \in E, \quad k \geq 0;$

ii) Let

$$N_n(t) = |a_{n,0}(t)| + |a_{n,1}(t)| + \cdots + |a_{n,n}(t)| + \cdots$$

Then there exists a function $R(t)$, such that:

$$N_n(t) < R(t), \quad R(t) < \infty, \quad t \in E, \quad n \geq 0.$$

iii) Let

$$A_n(t) = a_{n,0}(t) + a_{n,1}(t) + \cdots + a_{n,n}(t) + \cdots$$

Then

$$\lim_{n\to\infty} A_n(t) = 1, \ \ t \in E.$$

Summability method defined by matrix $M(t)$ we will call by $M(t)$ method. It is clear that $M(t)$ method will be regular method of summability i. e. if for a sequence of numbers S_k

$$\lim_{k\to\infty} S_k = S,$$

then, for the means

$$\tau_n(t) = \sum_{k=0}^{\infty} a_{n,k}(t) \cdot S_k$$

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of $M(t)$ method, we have

$$\lim_{n\to\infty} \tau_n(t) = S, \quad \text{for each} \quad t \in E.$$

For summability methods A and B, relation $A \subset B$ denots that if the sequence S_k is summable by method A to a number S, then this sequence is summable to the number S by method B.

Let $\{S_k(x)\}_{k=0}^{\infty}$ be sequence of functions defined on the set E. By $\tau_n(t,x)$ we denote the means of $M(t)$ method for $\{S_k(x)\}$ i. e.

$$\tau_n(t,x) = \sum_{k=0}^{\infty} a_{n,k}(t) \cdot S_k(x), \qquad (t,x) \in E^2 = E \times E.$$

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \tag{1}$$

be the Fourier series of an integrable function f.

By $S_n(x;f)$ we denote partial sums of the series (1), i. e.

$$S_n(x;f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

Let

$$\Lambda(t) = \|\lambda_{n,k}(t)\|$$

be a triangular matrix, constructed by measurable functions $\lambda_{n,k}(t)$ defined on $E = [0, 2\pi]$. Let for each $t \in [0, 2\pi]$ these functions satisfy the following conditions:

1) $\lambda_{n,0}(t) \equiv 1$ and $\lambda_{n,k}(t) = 0, \quad n \geq 0, \quad k > n$;
2) $0 \leq \lambda_{n,k+1}(t) \leq \lambda_{n,k}(t) \leq 1, \quad n \geq 0, \quad 0 \leq k \leq n$;
3) $\lim\limits_{n\to\infty} \lambda_{n,k}(t) = 1, \quad k \geq 0$.

We consider the means of $\Lambda(t)$ method for the series (1):

$$\begin{aligned}\tau_n(t,x,f) &= \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_{n,k}(t)(a_k \cos kx + b_k \sin kx) = \\ &= \sum_{k=0}^{n} \left(\lambda_{n,k}(t) - \lambda_{n,k+1}(t)\right) S_k(x;f)\end{aligned} \tag{2}$$

We denote $a_{n,k}(t) = \lambda_{n,k}(t) - \lambda_{n,k+1}(t)$.

It is clear that, $a_{n,k}(t)$ satisfy the Toeplitz conditions i)-iii). So $\Lambda(t)$ method is a regular summability method for each $t \in [0, 2\pi]$.

Let $\{\alpha_n(t)\}$ be a sequence of measurable functions, such that:

$$0 \leq \alpha_n(t) \leq 1 \quad \text{for each} \quad t \in [0, 2\pi] \;\text{ and }\; n \geq 0.$$

Let for each k, $o \leq k \leq n$ and $t \in [0, 2\pi]$, the functions $\lambda_{n,k}(t)$ be defined by the equality

$$\lambda_{n,k}(t) = \frac{A_{n-k}^{\alpha_n(t)}}{A_n^{\alpha_n(t)}}, \quad \text{where} \quad A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}. \qquad (3)$$

For the case (3) we denote by $\sigma_n^{\alpha_n(t)}(x; f)$ and $(C, \{\alpha_n(t)\})$ means (2) and $\Lambda(t)$ method respectively.

If $\alpha_n(t) = \alpha = const$ for each $n \geq 0$ and $t \in [0, 2\pi]$ and the relation (3) is fulfilled, then $\Lambda(t)$ summability is, in fact, the Cesaro summability (C, α). If $\alpha_n(t) \equiv 0$, then we have the convergence.

We will consider the case when

$$\lim_{n\to\infty} \alpha_n(t) = 0 \quad \text{for each} \quad t \in [0, 2\pi].$$

In this case, for any $\alpha > 0$, the following relations hold

$$(C, 0) \subset (C, \{\alpha_n(t)\}) \subset (C, \alpha).$$

In [1] and [2] we formulated some theorems for the $\Lambda(t)$ summability in case $\alpha_n(t) \equiv \alpha_n$ for each $t \in [0, 2\pi]$ and $\alpha_n \to 0$ as $n \to \infty$.

In particular, it was established in [1] a theorem (see [1], Theorem 4, a), from which follows:

Theorem A. *For any sequence of numbers $\alpha_n \to 0+$ as $n \to \infty$, there exists the continuous function $f(x)$, such that the sequence of means $\sigma_n^{\alpha_n}(x; f)$ diverges at some poinzt x_0.*

In [2] we have the following theorem which gives the positive answer to the problem posed by V. Temlyakov. Namely

Theorem B. *For any continuous function $f(x)$ there exists a sequence of numbers $\alpha_n \downarrow 0$, such that the equality*

$$\lim_{n\to\infty} \sigma_n^{\alpha_n}(x; f) = f(x)$$

holds at every point x.

In [1] we have a theorem (see [1], Theorem 4, b) one of corollaries of which is the following theorem about summability by norm L_1:

Theorem C. *For any sequence of numbers $\alpha_n \to 0+$ as $n \to \infty$ there exists a function $f \in L[0, 2\pi]$, such that*

$$\overline{\lim_{n\to\infty}} \|\sigma_n^{\alpha_n}(x; f) - f(x)\|_{L_1} > 0.$$

In the present talk we formulate analogous theorems for integrable functions. Namely there are established theorems for pointwise and L_1 norm summability for methods $(C, \{\alpha_n\})$ and $(C, \{\alpha_n(t)\})$. The following theorems are valid

Theorem 1. *For any function $f \in L_1[0, 2\pi]$ there exists a sequence of numbers $\alpha_n \downarrow 0$, such that*

$$\lim_{n\to\infty} \|\sigma_n^{\alpha_n}(x; f) - f(x)\|_{L_1} = 0$$

Theorem 2. *For any function $f \in L_1[0, 2\pi]$ and any number $\varepsilon > 0$, there exist the sequence of numbers $\alpha_n \downarrow 0$ and a set $F \subset [0, 2\pi]$ with $\mu F > 2\pi - \varepsilon$, such that the equality*

$$\lim_{n\to\infty} \sigma_n^{\alpha_n}(x; f) = f(x)$$

holds at every point $x \in F$.

The following theorem is on the summability of the series (1) by method $(C, \{\alpha_n(t)\})$. In particular for means $\sigma_n^{\alpha_n(t)}(x; f)$, when $t = x$, we have

Theorem 3. *For any function $f \in L_1[0, 2\pi]$ there exists a sequence of measurable functions $\{\alpha_n(t)\}$, such that*

$$\lim_{n\to\infty} \alpha_n(t) = 0, \quad t \in [0, 2\pi],$$

and at each Lebesgue point x of function f (i. e. a. e. on $[0, 2\pi]$) we have

$$\lim_{n\to\infty} \sigma_n^{\alpha_n(x)}(x; f) = f(x).$$

Various corollaries follow from above mentioned results. We shall present some of them.

The following proposition is corollary of Theorem 1.

Corollary 1. *For any function $f \in L_1[0, 2\pi]$, there exist a sequence of numbers $\alpha_k \downarrow 0$ and subsequence of natural numbers $n_k \uparrow \infty$, such that the equality*

$$\lim_{k\to\infty} \sigma_{n_k}^{\alpha_k}(x; f) = f(x)$$

holds almost everywhere on $[0, 2\pi]$.

Definition 1. We say that a continuous function $g(x)$ is more bad in the C-sense than a continuous function $f(x)$ if there exists a sequence of numbers $\alpha_n \downarrow 0$, such that:

$$\lim_{n\to\infty} \sigma_n^{\alpha_n}(x; f) = f(x) \text{ for any } x$$

and at the same time the sequence of means of the function $g(x)$

$$\{\sigma_n^{\alpha_n}(x; g)\}_{n=1}^{\infty}$$

diverges at same point x_0.

Definition 2. We say that an integrable function $g(x)$ is more bad in the L-sense than a integrable function $f(x)$, if there exists a sequence of numbers $\alpha_n \downarrow 0$, such that

$$\lim_{n\to\infty} \|\sigma_n^{\alpha_n}(x; f) - f(x)\|_L = 0$$

and at the same time

$$\overline{\lim_{n\to\infty}}\|\sigma_n^{\alpha_n}(x;g)-g(x)\|_{L_1}>0.$$

Corollary 2. *For any continuous function $f(x)$ there exists a continuous function $g(x)$ which is more bad in the C-sense than $f(x)$.*

Corollary 3. *For any continuous function $f_1(x)$, there exists a sequence of continuous functions*

$$f_1(x), f_2(x), \dots, f_{n-1}(x), f_n(x), \dots$$

such that each $f_n(x)$ is more bad in the C-sense then $f_{n-1}(x)$.

Corollary 2 and Corollary 3 follow from Theorem A and Theorem B.

The following propositions are consequences of Theorem C and Theorem 1.

Corollary 4. *For any integrable function $f(x)$ there exists integrable function $g(x)$ which is more bad in the L-sense than $f(x)$.*

Corollary 5. *For any integrable function $f_1(x)$, there exists a sequence of integrable functions*

$$f_1(x), f_2(x), \dots, f_{n-1}(x), f_n(x), \dots$$

such that each $f_n(x)$ is more bad in the L-sense then $f_{n-1}(x)$.

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References

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