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EXISTENCE OF OSCILLATING SOLUTION FOR THE INTEGRO-DIFFERENTIAL EQUATION

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1. Introduction

Consider the following boundary value problem

$$x'(t) = p(t) \int_0^{\sigma(t)} q(s)\, x(\tau(s))\, ds, \tag{1.1}$$

$$x(t) = \varphi(t) \quad \text{for} \quad t \in [\tau_0, 0), \quad \liminf_{t\to+\infty} \big|x(t)\big| < +\infty, \tag{1.2}$$

where

$$\begin{gathered} p \in C(R_+;(0,+\infty)), \quad q \in L_{\rm loc}(R_+;R_+), \quad \tau,\sigma \in C(R_+;R), \\ 0 \le \sigma(t) \le t, \quad \tau(t) < t \quad \text{for} \quad t \in R_+ \quad \text{and} \\ \lim_{t\to+\infty} \sigma(t) = \lim_{t\to+\infty} \tau(t) = +\infty, \end{gathered} \tag{1.3}$$

$$\begin{gathered} \tau_0 = \inf\big\{\tau(t) : t \in R_+\big\}, \quad \varphi \in C([\tau_0,0)) \quad \text{and} \\ \sup\big\{|\varphi(t)| : t \in [\tau_0,0)\big\} < +\infty. \end{gathered} \tag{1.4}$$

For any $\gamma \in R$, denote by $x(\cdot\,;\gamma)$ the solution of (1.1) satisfying the Cauchy problem

$$x(t) = \varphi(t) \quad \text{for} \quad t \in [\tau_0, 0), \quad \text{and} \quad x(0;\gamma) = \gamma. \tag{1.5}$$

A solution $x(\cdot\,;\gamma)$ of problem (1.1), (1.5) is said to be a proper solution if

$$\sup\big\{|x(s;\gamma)| : s \in [t,+\infty)\big\} > 0 \quad \text{for any} \quad t \in R_+.$$

A proper solution $x(\cdot\,;\gamma)$ of problem (1.1), (1.5) is said to be oscillatory if it has a sequence of zeros tending to $+\infty$. Otherwise the solution $x(\cdot\,;\gamma)$ is said to be nonoscillatory.

We obtain sufficient conditions for existence of solutions to problem (1.1), (1.2), its unique solvability and existence of unique oscillating solution.

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Analogous problem for the second order differential equation was considered in [1–3].

2. Existence of Solutions to Problem (1.1), (1.2)

Theorem 2.1. *Let the conditions* (1.3) *and* (1.4) *be fulfilled and*

$$\int_0^{+\infty} p(t)\,dt = +\infty, \tag{2.1}$$

$$\int_t^{+\infty} q(s)\,ds > 0 \quad \textit{for any} \quad t \in R_+. \tag{2.2}$$

Then problem (1.1), (1.2) *has at least one solution.*

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be fulfilled,*

$$\liminf_{t\to+\infty} \int_{\tau(\sigma(t))}^{t} q(\sigma(s))\,\sigma'(s)\,ds > 0, \tag{2.3}$$

$$\limsup_{t\to+\infty} \frac{1}{h(t)} \int_0^t q(\sigma(s))\,\sigma'(s)\,ds < +\infty, \tag{2.4}$$

$$\operatorname{vrai\,sup}\big(q(\sigma(t))\,\sigma'(t) : t \in R_+\big) < +\infty, \quad \inf\big\{p(t) : t \in R_+\big\} > 0, \tag{2.5}$$

$$\limsup_{t\to+\infty}\big(h(t) - h(\sigma(t))\big) < +\infty \tag{2.6}$$

and for any $\lambda \in R_+$

$$\limsup_{\varepsilon\to0+}\bigg(\liminf_{t\to+\infty} \exp\big((\lambda+\varepsilon)h(t)\big) \int_t^{+\infty} p(s)\times \\ \times \int_{\sigma(s)}^{+\infty} q(\xi)\,\exp\big(-(\lambda+\varepsilon)h(\tau(\xi))\big)d\xi\,ds\bigg) > 1, \tag{2.7}$$

where

$$h(t) = \int_0^t p(s)\,ds. \tag{2.8}$$

Then problem (1.1), (1.2) *has at least one solution with the sequence of zeros tending to infinity.*

Theorem 2.3. *Let the conditions of Theorem* 2.1*, inequalities* (2.3)*,* (2.6)*,* (2.7) *and the second inequality of* (2.5) *be fulfilled. If*

$$\operatorname{vrai\,sup}\big(h(t)\,q(\sigma(t))\,\sigma'(t) : t \in R_+\big) < +\infty, \tag{2.9}$$

$$\limsup_{t\to+\infty} \frac{1}{h(t)} \int_{t_1}^{t} h^{-1}\big(\tau(\sigma(s))\big)\,q(\sigma(s))\,\sigma'(s)ds < +\infty, \tag{2.10}$$

where h is defined by (2.8) *and t_1 is a sufficiently large number, then problem* (1.1)*,* (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Theorem 2.4. *Let the conditions of Theorem* 2.1*, inequalities* (2.6)*,* (2.7) *and the second inequality of* (2.5) *be fulfilled. If*

$$\liminf_{t\to+\infty} \int_{\tau(\sigma(t))}^{t} h(s)\,q(\sigma(s))\,\sigma'(s)\,ds > 0, \tag{2.11}$$

$$\limsup_{t\to+\infty} \frac{1}{h(t)} \int_{0}^{t} q(\sigma(s))\,\sigma'(s)\,h(s)\,ds < +\infty, \tag{2.12}$$

$$\operatorname{vrai\,sup}\big(h^2(t)\,q(\sigma(t))\,\sigma'(t) : t \in R_+\big) < +\infty, \tag{2.13}$$

where h is defined by (2.8)*, then problem* (1.1)*,* (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Corollary 2.1. *Let the conditions of Theorem* 2.1 *and inequalities* (2.3)*–*(2.6) *be fulfilled. If*

$$\inf\big(\lambda^{-2}\,a_p(\lambda)\,a_q(\lambda) : \lambda > 0\big) > 1,$$

where

$$a_p(\lambda) = \inf\big(e^{\lambda(h(t)-h(\sigma(t)))} : t \ge t_1\big),$$
$$a_q(\lambda) = \inf\left(\frac{q(t)}{p(t)}\,e^{\lambda(h(t)-h(\tau(t)))} : t \ge t_1\right),$$

t_1 is a sufficiently large number, then problem (1.1)*,* (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Corollary 2.2. *Let the conditions of Theorem* 2.1 *and inequalities* (2.3)*–*(2.6) *be fulfilled. If*

$$\inf\big(\lambda^{-2}\,a_p(\lambda)\,a_q(\lambda) : \lambda > 0\big) > 1,$$

where h is defined by (2.8) *and t_1 is great enough, then problem* (1.1)*,* (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Corollary 2.3. *Let $p(t) = p$, $q(t) = q$, $\tau(t) = t - \Delta$, $\sigma(t) = t$, where $p, q \in (0, +\infty)$, $\Delta > 0$. Then the condition*

$$\Delta \sqrt{pq} > \frac{2}{e}$$

is sufficient for problem (1.1), (1.2) *to have at least one solution with the sequence of zeros tending to $+\infty$.*

Theorem 2.5. *Let the conditions of Theorem* 2.1, *inequalities* (2.3), (2.5) *be fulfilled. If*

$$\limsup_{t\to+\infty} \frac{1}{\ln h(t)} \int_0^t q(\sigma(s))\,\sigma'(s)\,ds < +\infty, \tag{2.14}$$

$$\limsup_{t\to+\infty} \frac{h(t)}{h(\sigma(t))} < +\infty \tag{2.15}$$

and for any $\lambda \in R_+$

$$\limsup_{\varepsilon\to0+} \left(\liminf_{t\to+\infty} (h(t))^{\lambda+\varepsilon} \int_t^{+\infty} p(s) \int_{\sigma(s)}^{+\infty} q(\xi)\big(h(\tau(\xi))\big)^{-(\lambda+\varepsilon)} d\xi\,ds \right) > 1, \tag{2.16}$$

where h is defined by (2.8), *then problem* (1.1), (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Theorem 2.6. *Let the conditions of Theorem* 2.1, *inequalities* (2.11), (2.13), (2.15) *and* (2.16) *be fulfilled. If*

$$\limsup_{t\to+\infty} \frac{1}{\ln h(t)} \int_0^t h(s)\,q(\sigma(s))\,\sigma'(s)\,ds < +\infty,$$

then problem (1.1), (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Theorem 2.7. *Let the conditions of Theorem* 2.1, *inequalities* (2.9), (2.11), (2.15) *and* (2.16) *be fulfilled. If*

$$\limsup_{t\to+\infty} \frac{1}{\ln h(t)} \int_{t_1}^t h^{-1}(\sigma(s))\,q(\sigma(s))\,\sigma'(s)\,ds < +\infty,$$

where t_1 is a sufficiently large number, then problem (1.1), (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Corollary 2.4. *Let the conditions of Theorem* 2.1, *inequalities* (2.3), (2.5), (2.14) *and* (2.15) *be fulfilled. If*

$$\inf \left(\frac{1}{\lambda(1+\lambda)}\, a_p(\lambda)\, a_q(\lambda) : \lambda > 0 \right) > 1,$$

where

$$a_p(\lambda) = \inf\left(\left(\frac{h(t)}{h(\sigma(t))}\right)^{1+\lambda} : t \geq t_1\right),$$

$$a_q(\lambda) = \inf\left(\frac{q(t)}{p(t)}\, h^2(t)\left(\frac{h(t)}{h(\tau(t))}\right)^{\lambda} : t \geq t_1\right),$$

t_1 *is a sufficiently large number, then problem* (1.1), (1.2) *has at least one solution with the sequence of zeros tending to* $+\infty$.

Corollary 2.5. *Let* $p(t) = p$, $q(t) = \frac{q}{(1+t)^2}$, $\sigma(t) = \alpha t$, $\tau(t) = \beta(t - \Delta)$, *where* $p, q, \Delta \in (0, +\infty)$, $\alpha, \beta \in (0, 1)$. *Then the condition*

$$\inf\left(\frac{1}{\lambda(1+\lambda)}\, \alpha^{-\lambda-1}\, \beta^{-1} : \lambda \in (0, +\infty)\right) > \frac{1}{pq}$$

is sufficient for problem (1.1), (1.2) *to have at least one solution with the sequence of zeros tending to* $+\infty$.

Theorem 2.8. *Let the conditions of Theorem* 2.1, *inequalities* (2.3) *and* (2.5) *be fulfilled. If*

$$\limsup_{t\to+\infty} \frac{1}{\ln(lnh(t))} \int_0^t q(\sigma(s))\, \sigma'(s)\, ds < +\infty, \tag{2.17}$$

$$\limsup_{t\to+\infty} \frac{\ln h(t)}{\ln\ln(\sigma(t))} < +\infty, \tag{2.18}$$

and for any $\lambda \in R_+$

$$\limsup_{t\to+\infty}\Bigg(\liminf_{t\to+\infty}\, \big(\ln h(t)\big)^{\lambda+\varepsilon} \inf_t^{+\infty} p(s) \times \\ \times \int_{\sigma(s)}^{+\infty} q(\xi)\big(\ln h(\tau(\xi))\big)^{-(\lambda+\varepsilon)} d\xi\, ds\Bigg) > 1, \tag{2.19}$$

then problem (1.1), (1.2) *has at least one solution with the sequence of zeros tending to* $+\infty$.

Theorem 2.9. *Let the conditions of Theorem* 2.1, *inequalities* (2.3), (2.9), (2.18) *and* (2.19) *be fulfilled. If*

$$\limsup_{t\to+\infty} \frac{1}{\ln\ln h(t)} \int_{t_1}^t h^{-1}(\tau(\sigma(s)))\, q(\sigma(s))\, \sigma'(s)\, ds < +\infty,$$

where t_1 *is sufficiently large number, then problem* (1.1), (1.2) *has at least one solution with the sequence of zeros tending to* $+\infty$.

Theorem 2.10. *Let the conditions of Theorem* 2.1*, inequalities* (2.11), (2.13)*,* (2.18) *and* (2.19) *be fulfilled. If*

$$\limsup_{t\to+\infty} \frac{1}{\ln\ln t}\int\limits_0^t h(s)\,q(\sigma(s))\,\sigma'(s)\,ds < +\infty,$$

where h is defined by (2.8)*, then problem* (1.1)*,* (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

Corollary 2.6. *Let the conditions of Theorem* 2.1*, inequalities* (2.11), (2.13)*,* (2.17) *and* (2.18) *be fulfilled. If*

$$\inf\left(\frac{1}{\lambda}\,a_p(\lambda)\,a_q(\lambda) : \lambda > 0\right) > 1,$$

where

$$a_p(\lambda) = \inf\left(\frac{h(t)}{h(\sigma(t))}\left(\frac{\ln h(t)}{\ln h(\sigma(t))}\right)^{1+\lambda} : t \ge t_1\right),$$

$$a_q(\lambda) = \inf\left(\frac{q(t)\,h^2(t)\,\ln h(t)}{p(t)}\,h^2(t)\left(\frac{\ln h(t)}{\ln h(\tau(t))}\right)^{\lambda} : t \ge t_1\right),$$

t_1 is a sufficiently large number, then problem (1.1)*,* (1.2) *has at least one solution with the sequence of zeros tending to $+\infty$.*

3. The Existence of Oscillatory Solutions

Theorem 3.1. *Assume that*

$$\varphi(\xi) > 0 \quad \text{for} \quad \xi \in [\tau_0, 0), \tag{3.1}$$

inequality (2.2) *and the conditions at least one of Theorems* 2.2–2.10 *are fulfilled. Then problem* (1.1)*,* (1.2) *has at least one oscillating solution.*

Theorem 3.2. *Let conditions* (1.3)*,* (1.4)*,* (2.1) *and* (2.2) *be fulfilled, σ be a nondecreasing function and*

$$\limsup_{t\to+\infty} \int\limits_t^{\eta(t)} p(s) \int\limits_{\sigma(s)}^{\sigma(\eta(s)} q(\xi)d\xi\,ds < +\infty.$$

Then problem (1.1)*,* (1.2) *has only one solution, where*

$$\eta(t) = \sup\big\{s : \tau(\sigma(s)) < t\big\}.$$

Theorem 3.3. *Let conditions of the Theorems* 3.1 *and* 3.2 *be fulfilled. Then problem* (1.1)*,* (1.2) *has the unique oscillating solution.*

Corollary 3.1. *Let the conditions of Corollary* 2.3 *hold. Then problem* (1.1)*,* (1.2) *has the unique oscillating solution for any φ satisfying condition* (3.1)*.*

Corollary 3.2. *Let the conditions of Corollary* 2.5 *be fulfilled. Then problem* (1.1), (1.2) *has the unique oscillating solution for any* φ *satisfying condition* (3.1).

Corollary 3.3. *Let* $p(t) = p$, $q(t) = \dfrac{q}{(1+t)^2 \ln(2+t)}$, $\sigma(t) = t$ *and* $\tau(t) = (t-\Delta)^\alpha$, *where* $\Delta > 0$, $\alpha \in (0,1)$. *Then the inequality*

$$pq|\ln \alpha| > \frac{1}{e}$$

is sufficient for problem (1.1), (1.2) *to have a unique oscillating solution for any* φ *satisfying condition* (3.1).

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