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Some Properties of the Semigroup $B_X(D)$ Determined by a Semilattice of Class $\Sigma_1(X, 4)$

A binary relation on a set X is a subset of the cartesian product $X \times X$. If α and β are binary relations on X , their product $\alpha \circ \beta$ is defined in the following way: $(x, y) \in \alpha \circ \beta$ ($x, y \in X$), if there exists an element $z \in X$, such that $(x, z) \in \alpha$ and $(z, y) \in \beta$.

If Y is some subset of X and α is some binary relation on X , then by $Y\alpha$ we will denote the collection of all such elements $x \in X$ for which there can be found an $y \in Y$ such that $(y, x) \in \alpha$.

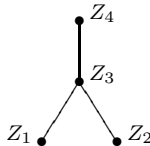
Let D be some set of subsets of X closed under the operations of set-theoretic union of elements from D , i.e., $\bigcup D' \in D$ for any non-empty subset D' of D . In such case D is called a complete join X -semilattice. By $V(D, \alpha)$ will be denoted the following set: $V(D, \alpha) = \{Y\alpha | Y \in D\}$.

Let X be any non-empty set and D be a complete join X -semilattice. Let f be any map of the set X to the set D . To each such map let us assign the binary relation α_f on the set X satisfying the condition $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$. The set of all such α_f ($f : X \rightarrow D$) will be denoted $B_X(D)$. It can be proved, that $B_X(D)$ is a subsemilattice of the semilattice B_X of all binary relations on the set X . The semilattice $B_X(D)$ is called the complete semigroup of binary relations determined by the complete join X -semilattice D [1-3].

Let X be any non-empty set and $D = \{Z_1, Z_2, Z_3, Z_4\}$ be the set of subsets of the set X , where the elements Z_1, Z_2, Z_3, Z_4 satisfy the following conditions:

$$Z_1 \subset Z_3, Z_2 \subset Z_3, Z_1 \cup Z_2 = Z_3, Z_3 \subset Z_4 \text{ and } Z_1 \cap Z_2 = \emptyset \quad (1)$$

(notation $Z \subset Z'$ means that $Z \subseteq Z'$ and $Z \neq Z'$). Obviously, D is a complete join X -semilattice, and its corresponding diagram is of shape



The class of all semilattices $D = \{Z_1, Z_2, Z_3, Z_4\}$, whose elements satisfy the conditions (1), will be denoted by $\Sigma_1(X, 4)$. The properties (1) imply that $\emptyset \notin D$ and $|X| \geq 4$. Note also that each element α of the semigroup $B_X(D)$ determined by a complete X -semilattice of class $\Sigma_1(X, 4)$ can be uniquely represented in the following way: $\alpha = (Y_1 \times Z_1) \cup (Y_2 \times Z_2) \cup (Y_3 \times Z_3) \cup ((X \setminus (Y_1 \cup Y_2 \cup Y_3)) \times Z_4)$, where $Y_i \subseteq X$, $Y_j \cap Y_k = \emptyset$ for $j \neq k$ ($i, j, k = 1, 2, 3$).

Theorem 1. *Let X be a finite set of cardinality $n \geq 4$. Then the number of elements in the class $\Sigma_1(X, 4)$ is computed by the formula*

$$|\Sigma_1(X, 4)| = \frac{1}{2} - 2^{n+1} + 3^{n+1} - 2^{2n+1} + \frac{1}{2} \cdot 5^n.$$

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Theorem 2. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then $|V(D, \alpha)| \leq 3$ for every element $\alpha \in B_X(D)$.

Corollary 1. The semigroup $B_X(D)$ determined by a semilattice of class $\Sigma_1(X, 4)$ does not have left units.

Definition 1. A binary relation α from the semigroup $B_X(D)$ is called an idempotent element if $\alpha \circ \alpha = \alpha$.

Theorem 3. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then the binary relation α from the semigroup $B_X(D)$ is an idempotent element of this semigroup if and only if it satisfies one of the following conditions:

- a) $\alpha = X \times Z_i$, where $i = 1, 2, 3, 4$,
- b) $\alpha = (Y \times Z_i) \cup ((X \setminus Y) \times Z_j)$, where $i = 1, 2, 3$, $j = 3, 4$, $i < j$ (i.e., $Z_i \subset Z_j$), $Y \subset X$ and satisfies the conditions $Y \supseteq Z_i$ and $Z_j \setminus Y \neq \emptyset$,
- c) $\alpha = (Y_1 \times Z_i) \cup (Y_2 \times Z_3) \cup ((X \setminus (Y_1 \cup Y_2)) \times Z_4)$, where $i = 1, 2$, $Y_1, Y_2 \subset X$ and satisfy the conditions $Y_1 \cap Y_2 = \emptyset$, $Y_1 \supseteq Z_i$, $Y_1 \cup Y_2 \supseteq Z_3$, $Y_2 \cap Z_3 \neq \emptyset$ and $Z_4 \setminus (Y_1 \cup Y_2) \neq \emptyset$.

Corollary 2. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then the number of elements in the set I of all idempotent elements of the finite semigroup $B_X(D)$ is computed by the formula

$$|I| = 4 + 2^{|X \setminus Z_1|+1} + 2^{|X \setminus Z_2|+1} - 2^{|X \setminus Z_3|} - 3 \cdot 2^{|X \setminus Z_4|} + \\ + 3^{|X \setminus Z_4|} \cdot (3^{|Z_4 \setminus Z_3|} - 2^{|Z_4 \setminus Z_3|}) \cdot (2^{|Z_3 \setminus Z_1|} + 2^{|Z_3 \setminus Z_2|} - 2).$$

Note that the set I of all idempotent elements of the semigroup $B_X(D)$ determined by a semilattice of class $\Sigma_1(X, 4)$ does not form a subsemigroup of the semigroup $B_X(D)$.

Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Obviously the sets $D_1 = \{Z_1, Z_3, Z_4\}$, $D_2 = \{Z_2, Z_3, Z_4\}$ and $D_{ij} = \{Z_i, Z_j\}$, where $i = 1, 2, 3$; $j = 3, 4$; $i < j$ (i.e., $Z_i \subset Z_j$) are complete X -semilattices, hence it makes sense to consider the complete semigroups of binary relations $B_X(D_1)$, $B_X(D_2)$ and $B_X(D_{ij})$ determined respectively by the semilattices $D_1 = \{Z_1, Z_3, Z_4\}$, $D_2 = \{Z_2, Z_3, Z_4\}$ and $D_{ij} = \{Z_i, Z_j\}$.

Denote by $E_X^{(r)}(D_1)$ and $E_X^{(r)}(D_2)$ respectively the sets of all those idempotent elements of the semigroups $B_X(D_1)$ and $B_X(D_2)$ which satisfy the condition c) of theorem 3, whereas by I_{ij} will be denoted the set of all idempotent elements of the semigroup $B_X(D_{ij})$ ($i = 1, 2, 3$, $j = 3, 4$, $i < j$). By $K_X(D)$ is denoted the set of all elements of the form $X \times Z_i$ ($i = 1, 2, 3, 4$). Let us also introduce the following notations:

$$I'_{14} = \{\alpha \in I_{14} | \alpha = (Y \times Z_1) \cup ((X \setminus Y) \times Z_4) : Z_3 \setminus Y \neq \emptyset\}, \\ I''_{14} = \{\alpha \in I_{14} | \alpha = (Y \times Z_1) \cup ((X \setminus Y) \times Z_4) : Z_3 \subseteq Y\}, \\ I'_{24} = \{\alpha \in I_{24} | \alpha = (Y \times Z_2) \cup ((X \setminus Y) \times Z_4) : Z_3 \setminus Y \neq \emptyset\} \text{ and} \\ I''_{24} = \{\alpha \in I_{24} | \alpha = (Y \times Z_2) \cup ((X \setminus Y) \times Z_4) : Z_3 \subseteq Y\}.$$

Obviously $X \times Z_1 \in I'_{14}$, $X \times Z_4 \in I'_{14}$, $X \times Z_2 \in I'_{24}$ and $X \times Z_4 \in I'_{24}$.

Theorem 4. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then all maximal subsemigroups of idempotent elements of the semigroup $B_X(D)$ are exhausted by the following subsemigroups of the semigroup $B_X(D)$:

- a) $I_1 = E_X^{(r)}(D_1) \cup E_X^{(r)}(D_2) \cup I_{13} \cup I_{23} \cup I_{34}$,
- b) $I_2 = E_X^{(r)}(D_1) \cup I_{13} \cup I_{23} \cup I'_{14}$,
- c) $I_3 = E_X^{(r)}(D_1) \cup E_X^{(r)}(D_2) \cup I_{23} \cup I_{34} \cup I''_{14}$,
- d) $I_4 = E_X^{(r)}(D_2) \cup I_{13} \cup I_{23} \cup I'_{24}$,
- e) $I_5 = E_X^{(r)}(D_1) \cup E_X^{(r)}(D_2) \cup I_{13} \cup I_{34} \cup I''_{24}$,
- f) $I_6 = E_X^{(r)}(D_1) \cup I_{23} \cup I_{14}$,

- g) $I_7 = E_X^{(r)}(D_2) \cup I_{13} \cup I_{24}$,
 h) $I_8 = I_{13} \cup I_{23} \cup I'_{14} \cup I'_{24}$,
 i) $I_9 = E_X^{(r)}(D_1) \cup E_X^{(r)}(D_2) \cup I_{34} \cup I''_{14} \cup I''_{24}$.

Theorem 5. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$, $\beta_{13} = (Z_1 \times Z_1) \cup ((X \setminus Z_1) \times Z_3)$, $\beta_{23} = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_3)$, $\beta'_{14} = (Z_1 \times Z_1) \cup ((X \setminus Z_1) \times Z_4)$, $\beta'_{14} = (Z_3 \times Z_1) \cup ((X \setminus Z_3) \times Z_4)$, $\beta'_{24} = (Z_2 \times Z_2) \cup ((X \setminus Z_2) \times Z_4)$, $\beta''_{24} = (Z_3 \times Z_2) \cup ((X \setminus Z_3) \times Z_4)$ and $B = \{\beta_{13}, \beta_{23}, \beta'_{14}, \beta'_{14}, \beta'_{24}, \beta''_{24}\}$. Then the set $I' = B \cup E_X^{(r)}(D_1) \cup E_X^{(r)}(D_2)$ is a unique irreducible generating set of the subsemigroup $I_X(D)$ generated by the set of all idempotent elements of the semigroup $B_X(D)$.

Definition 2. A binary relation α from the semigroup $B_X(D)$ is called a regular element if there exists an element $\beta \in B_X(D)$ such that $\alpha \circ \beta \circ \alpha = \alpha$.

Theorem 6. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then a binary relation α from the semigroup $B_X(D)$ is a regular element of this semigroup if and only if it satisfies one of the following conditions:

- a) $\alpha = X \times Z_i$, where $i = 1, 2, 3, 4$,
 b) $\alpha = (Y \times Z_i) \cup ((X \setminus Y) \times Z_j)$, where $i = 1, 2, 3$, $j = 3, 4$, $i < j$ (i.e., $Z_i \subset Z_j$), $Y \subset X$ and satisfies the conditions $Z_j \setminus Y \neq \emptyset$ and $Y \supseteq Z$, where $Z \in \{Z_1, Z_2, Z_3\}$,
 c) $\alpha = (Y_1 \times Z_i) \cup (Y_2 \times Z_3) \cup ((X \setminus (Y_1 \cup Y_2)) \times Z_4)$, where $i = 1, 2$, $Y_1, Y_2 \subset X$ and satisfy the following conditions: $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 \supseteq Z_3$, $Y_2 \cap Z_3 \neq \emptyset$, $Z_4 \setminus (Y_1 \cup Y_2) \neq \emptyset$ and $Y_1 \supseteq Z$, where $Z \in \{Z_1, Z_2\}$.

Corollary 3. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then the number of elements in the set R of all regular elements of the finite semigroup $B_X(D)$ is computed by the formula

$$|R| = 4 + 5 \cdot (2^{|X \setminus Z_1|} + 2^{|X \setminus Z_2|} - 2^{|X \setminus Z_3|} - 2^{|X \setminus Z_4|}) + \\ + 3^{|X \setminus Z_4|} \cdot (3^{|X \setminus Z_4|} - 2^{|X \setminus Z_4|}) \cdot (2^{|Z_3 \setminus Z_1|+1} + 2^{|Z_3 \setminus Z_2|+1} - 4).$$

Definition 3. A binary relation α from the semigroup $B_X(D)$ is called an externally adjoined element if $\beta \circ \delta \neq \alpha$ for any $\beta, \delta \in B_X(D) \setminus \{\alpha\}$.

Theorem 7. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then a binary relation α from the semigroup $B_X(D)$ is an externally adjoined element of this semigroup if and only if it satisfies one of the following conditions:

- a) $\alpha = (Y_1 \times Z_1) \cup ((X \setminus Y_1) \times Z_2)$,
 b) $\alpha = (Y_1 \times Z_1) \cup (Y_2 \times Z_2) \cup ((X \setminus (Y_1 \cup Y_2)) \times Z_3)$,
 c) $\alpha = (Y_1 \times Z_1) \cup (Y_2 \times Z_2) \cup ((X \setminus (Y_1 \cup Y_2)) \times Z_4)$,
 d) $\alpha = (Y_1 \times Z_1) \cup (Y_2 \times Z_2) \cup (Y_3 \times Z_3) \cup ((X \setminus (Y_1 \cup Y_2 \cup Y_3)) \times Z_4)$,
 where $Y_1, Y_2, Y_3 \subset X$, $Y_i \cap Y_j = \emptyset$ for $i \neq j$ ($i, j = 1, 2, 3$), $Y_1, Y_2, Y_3 \notin \{\emptyset, X\}$ and $Y_1 \cup Y_2 \neq X$.

Corollary 4. Let $D = \{Z_1, Z_2, Z_3, Z_4\} \in \Sigma_1(X, 4)$. Then the number of elements in the set S of all externally adjoined elements of the finite semigroup $B_X(D)$ is computed by the formula

$$|S| = 2^{|X|} - 2 \cdot 3^{|X|} + 4^{|X|}.$$

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