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A Topological Interpretation of Provability Predicate in Peano Arithmetic and Zermelo–Fraenkel Set Theory

We shall try to take the risk of a brief description of topological completeness of certain systems of Modal Logic that are of special interest in connection with the study of the notions of Provability in Peano Arithmetic and truth in transitive models for the Zermelo–Fraenkel Set Theory.

Recall that the modal system **K4** is the basic Modal Logic whose modal axioms are all expressions of the form (1) $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ and (2) $\Box p \rightarrow \Box \Box p$. The diamond $\Diamond$ as usual means the dual $\neg \Box \neg$ of $\Box$.

It is appropriate to mention here that axioms of the system **K4** are modal variants of the Hilbert–Bernays Derivability Conditions, i. e. **K4** axiomatizes these properties of the Proof Predicate $\text{Bew}(\cdot)$ that do not depend on the Gödel’s Diagonal Lemma.

Recall that X is said to be a T_d -space if every singleton subset of X is an intersection of an open and a closed subset. A fundamental topological operation is that of the *derivative* (alias, *limit*) of a point set A , which will be denoted by $\mathbf{d}A$. The set $\mathbf{d}A$ is, by definition, the set of all limit (alias, accumulation) points of a subset A of a topological space X . A point x is said to be a *limit point* of a set A , if every neighborhood of x contains a point of A other than x .

Let X be a topological space. Every valuation mapping propositional variables to subsets of X can be extended to a *topological interpretation* $(\cdot)^*$, assigning subsets of X to all formulas of a modal system by the inductive definition: $(p \vee q)^* = (p^* \cup q^*)$, $(p \wedge q)^* = (p^* \cap q^*)$, $(\neg p)^* = X - p^*$ and $(\Diamond p)^* = \mathbf{d}(p^*)$, where $\mathbf{d}$ is the limit-operation of the space X . A formula p is valid in a space X if $p^* = X$ for every topological interpretation $*$.

Proposition 1. *Topological completeness of **K4**:*

K4 $\vdash p$ iff the formula p is valid in every T_d -space.

The modal system **K4** is evidently arithmetically sound but it is not complete. An example of a theorem of **PA** not provable in **K4** is $\neg \Box \perp \rightarrow \neg \Box \neg \Box \perp$, a modal version of the famous discovery of Gödel (the so called Second Incompleteness Theorem) that if Peano Arithmetic is consistent then consistency of **PA** is not provable in **PA**.

Let **K4*** denote the system that results when the schema $\neg \Box \perp \rightarrow \neg \Box \neg \Box \perp$ is added to the modal system **K4**.

Proposition 2. *Topological completeness of **K4***:*

K4* $\vdash p$ iff the formula p is valid in every T_d -space X , in which the set of all isolated points is a dense subset of X .

The Gödel Modal System **GL** (alias, the *Provability Logic*) is of interest since it adequately reflects behavior of the Provability Predicate in Peano Arithmetic **PA**. **GL** is the result of adding the axiom $\Box(\Box p \rightarrow p) \rightarrow \Box p$ to the system **K4**. Solovay [1] defines an arithmetical realization of a formula of the system **GL** (by reading the box-modality $\Box$ as “it is provable in **PA**”) and proves arithmetical completeness of **GL** (see also [2]).

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Definition (Cantor). A topological space is called a *scattered space* if it has no dense-in-itself nonempty subset.

An intuitionistic consideration of scatteredness can be found in [3].

An “equational” (à la Kuratowski) definition of scattered spaces using the limit operation as a primitive notion is contained in

Proposition 3. *A scattered space is the same as a set X equipped with an operator $\mathbf{d}$, satisfying the equations: 1) $\mathbf{d}(\emptyset) = \emptyset$, 2) $\mathbf{d}(A \cup B) = \mathbf{d}A \cup \mathbf{d}B$ and 3) $\mathbf{d}A = \mathbf{d}(A - \mathbf{d}A)$ for any subsets A and B .*

Corollary. *Topological completeness of $\mathbf{GL}$:*

$\mathbf{GL} \vdash \mathbf{p}$ iff p^* is valid in every scattered space.

In [1] Solovay presented also a set-theoretical interpretation of modal formulas by putting $\Box p$ to mean “ p is true in every transitive model of Zermelo-Fraenkel Set Theory $\mathbf{ZF}$ ”, formulated a modal system (which we call here $\mathbf{SL}$) and announced its $\mathbf{ZF}$ -completeness (for a proof, see [2], Chapt.13, p.166). The modal system $\mathbf{SL}$ results from $\mathbf{GL}$ by adding the formula $\Box(\Box p \rightarrow \Box q) \vee \Box(\Box q \rightarrow p)$ as a new axiom.

A Heyting lattice $\mathbf{H}$ is called a Boolean Cascade if there exist Boolean lattices $\mathbf{B}_1, \dots, \mathbf{B}_k$, such that $\mathbf{H} = \mathbf{B}_1 + \mathbf{B}_2 + \dots + \mathbf{B}_k$, where each $\mathbf{B}_i$ is a convex sublattice of $\mathbf{H}$ and $\mathbf{B}_i + \mathbf{B}_{i+1}$ denotes the ordinal sum of $\mathbf{B}_i$ and $\mathbf{B}_{i+1}$ in which the largest element of $\mathbf{B}_i$ and the smallest element of $\mathbf{B}_{i+1}$ are identified.

Proposition 4. *Topological completeness of $\mathbf{SL}$:*

$\mathbf{SL} \vdash \mathbf{p}$ iff p^* is valid in every scattered space X in which the lattice $Op(X)$ of all open sets is a Boolean cascade.

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