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Irreducible Elements of the Semigroup $B_X(D)$

Irreducible elements of the semigroup B_X were studied in [1], assuming that X is a finite set. In this paper, for the finite set X we establish several classes of irreducible elements of the semigroup $B_X(D)$ (see [2], Chapter 13).

In what follows $\bar{D}$ is assumed to be a nonempty set of pairwise nonintersecting subsets of the set X , which generates the complete join X -semilattice D ; $\check{D} = \cup D$, $\bar{D}^* = \bar{D} \setminus \{\emptyset\}$ and $\bar{D}^* \neq \emptyset$.

Definition 1. To each element T from the set $\bar{D}^*$ we assign some nonempty subset $\bar{T}$ of the set T . The set $\bar{D} = \{\bar{T} \mid T \in \bar{D}^*\}$ is called the system of representatives of elements from $\bar{D}^*$.

The set of all systems of representatives of elements from $\bar{D}^*$ is denoted by $\bar{\Omega}$. It is obvious that $\bar{D}^* \in \bar{\Omega}$. Now, for each $\bar{D} \in \bar{\Omega}$, let us arbitrarily fix some one-to-one mapping $g_{\bar{D}^* \bar{D}}$ of the set $\bar{D}^*$ onto the set $\bar{D}$. The set of all such mappings is denoted by $\bar{M}$. It is obvious that the set $\bar{M}$ is defined in a nonunique manner. Further, the set of all one-to-one mappings of the set $\bar{D}$ onto the set $\bar{D}^*$ is denoted by $G_{\bar{D} \bar{D}^*}$ and

$$G_{\bar{D}^*}(\Omega') = \bigcup_{\bar{D} \in \Omega'} G_{\bar{D} \bar{D}^*},$$

where $\emptyset \neq \Omega' \subseteq \bar{\Omega}$. Next, assume that f_1 and f_2 are arbitrary elements of the set $G_{\bar{D}^*}(\Omega')$. In the set $G_{\bar{D}^*}(\Omega')$ we define the binary relation $(*)$ as follows: $f_1 * f_2 = f_1 \cdot g_{\bar{D}^* \bar{D}} \cdot f_2$ if $f_1 \in G_{\bar{D} \bar{D}^*}$ ($g_{\bar{D}^* \bar{D}} \in \bar{M}$). In the right-hand side of the latter equality, $(\cdot)$ denotes the usual operation of multiplication of mappings. Clearly, the binary operation $(*)$ depends on a choice of the set $\bar{M}$. It is easy to verify that the set $G_{\bar{D}^*}(\Omega')$, where $\emptyset \neq \Omega' \subseteq \bar{\Omega}$, is a semigroup with respect to the binary operation $(*)$. Hence, in particular, for $\Omega' = \{\bar{D}\}$, where $\bar{D} \in \bar{\Omega}$, we obtain that the set $G_{\bar{D}^*}(\Omega') = G_{\bar{D} \bar{D}^*}$ is also a semigroup with respect to the operation $(*)$. Further, to each element f of the group $G_{\bar{D} \bar{D}^*}$ and to an arbitrary mapping f' of the set $X \setminus \check{D}$ in the semilattice D we assign the binary relations

$$\alpha(f; f') = \bigcup_{\bar{T} \in \bar{D}} (\bar{T} \times f(\bar{T})) \bigcup_{t \in X \setminus \check{D}} (\{t\} \times f'(t)).$$

The set of all such binary relations $\alpha(f; f')$ from $B_X(D)$, which corresponds to the pair of mappings f and f' , is denoted by $G'_X(\bar{D}; \bar{D}; D)$ and

$$G'_X(D; \Omega') = \bigcup_{\bar{D} \in \Omega'} G'_X(\bar{D}; \bar{D}; D),$$

where $\emptyset \neq \Omega' \subseteq \bar{\Omega}$. Note that for $\Omega' = \bar{\Omega}$ the set $G'_X(D; \Omega')$ is denoted by $G'_X(D)$. It obviously follows that if $\emptyset \notin \bar{D}$, then the conditions $\alpha \in B_X(D)$ always imply $t\alpha \neq \emptyset$ for all $t \in X$. Hence we conclude that the semigroup $G'_X(\bar{D}; \bar{D}; D)$, where $\bar{D} \in \bar{\Omega}$, is a subsemigroup of the semigroup $B_X(D)$ if and only if $\bar{D} = \bar{D}$. Therefore in that case the equality $G'_X(D) = G'_X(\bar{D}; \bar{D}; D)$ is fulfilled.

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Definition 2. We say that a binary relation α from $B_X(D)$ is irreducible in the semigroup $B_X(D)$ if for any representation of the relation α in the form $\alpha = \beta \circ \delta$, where $\beta, \delta \in B_X(D)$, we always have either $\beta \in G'_X(D)$ or $\delta \in G'_X(D)$.

Theorem 1. Let $\tilde{D}$ be a finite nonempty set of pairwise nonintersecting subsets of the set X , which generates the complete join X -semilattice D and $\beta \in G'_X(D)$. Then the binary relation β is irreducible in the semigroup $B_X(D)$.

Definition 3. Let $\tilde{D}$ be a set whose power is greater than or equal to two. We say that f is a cyclic bijection of the set $\tilde{D}$ if f is a one-to-one mapping of the set $\tilde{D}$ onto the set $\tilde{D}$ which satisfies the condition $D'f \neq D'$ for any and nonempty and proper subset D' from $\tilde{D}$, where $D'f$ denotes the set of all images of elements from D' of the mappings f .

Definition 4. Let $|\tilde{D}| \geq 3$. We say that a binary relation β from the semigroup $B_X(D)$ is closed on the set $\check{D} = \bigcup \tilde{D}$ if there exist a cyclic bijection f of the set $\tilde{D}^*$, a mapping f' of the set $X \setminus \check{D}$ in the semilattice D and $\bar{D} \in \bar{\Omega}$ such that

$$\beta = \beta(\bar{D}; f; f') = \bigcup_{T \in \tilde{D}^*} \left(g_{\bar{D}^* \bar{D}}(T) \times \left(T \bigcup f(T) \right) \right) \bigcup \bigcup_{t \in X \setminus \check{D}} (\{t\} \times f'(t)),$$

where $g_{\bar{D}^* \bar{D}} \in \bar{M}$, and the conditions $\emptyset \neq \bar{T} \subseteq T$, $\bar{T} \in \bar{D}$ and $g_{\bar{D}^* \bar{D}}(T) = \bar{T}$ are fulfilled.

Theorem 2. Let $|\tilde{D}^*| = n \geq 3$ and a relation β from $B_X(D)$ be a closed binary relation on the set $\check{D} = \bigcup \tilde{D}$. Then the following statements are valid for an arbitrary irreducible generating set D' of the semilattice $V(D; \beta) \setminus \{\emptyset\}$:

- 1) $|D'| \geq |\tilde{D}^*|$;
- 2) if D' is an n -element generating set of the semilattice $V(D; \beta) \setminus \{\emptyset\}$, then $D' = \tilde{D}^*$ or $D' = \tilde{D}^*(\beta)$ ($\tilde{D}^*(\beta) = \{t\beta \mid t \in \check{D} \setminus \{\emptyset\}\}$).

Theorem 3. Let $\tilde{D}$ be a finite set of pairwise nonintersecting subsets of the set X , which generates the complete join X -semilattice D and $|\tilde{D}^*| = n \geq 3$. If $\beta \in B_X(D)$ and β is a closed binary relation on the set $\check{D} = \bigcup \tilde{D}$, then β is irreducible in the semigroup $B_X(D)$.

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