

YA. DIASAMIDZE AND IL. DIASAMIDZE

Semigroups $B_X(D)$ Defined by Finite X -Chains

Let X be an arbitrary nonempty set. Recall that a nonempty set D of subsets from X is called an X -chain if D is a chain with respect to set-theoretic inclusion. In what follows we will consider those complete semigroups of binary relations $B_X(D)$ which are defined by finite X -chains (see [1]). Further, for $\emptyset \in D$ we denote by $C_0(s; D)$ the set of all s -element ($1 \leq s \leq |D|$) subchains D' of the chain D which satisfy the condition $\emptyset \in D'$, whereas for $\emptyset \notin D'$ we denote by $C_1(s; D)$ the set of all s -element subchains of the chain D . In addition to this, we denote by $\check{D} \in D''$ ($\check{D} = \cup D$) is fulfilled.

Let now $D = \{T_0, T_1, \dots, T_m\}$ ($m \geq 1$) be a finite X -chain having a representation of the form $T_0 \subset T_1 \subset \dots \subset T_m$, and let the chains $D' = \{T_{i_1}, T_{i_2}, \dots, T_{i_s}\} \in C_0(s; D)$ and $D'' = \{T_{j_1}, T_{j_2}, \dots, T_{j_s}\} \in C_1(s; D)$ satisfy the conditions

$$T_{i_1} \subset T_{i_2} \subset \dots \subset T_{i_s} \quad \text{and} \quad T_{j_1} \subset T_{j_2} \subset \dots \subset T_{j_s}.$$

For each k ($k = 2, 3, \dots, s$) we define the mappings $f_{j_k j_{k-1}}$ of the set $T_{j_k} \setminus T_{j_{k-1}}$ in the k -element set $D'_k = \{T_{i_1}, T_{i_2}, \dots, T_{i_k}\}$ as follows:

- a) $f_{j_k j_{k-1}}(t_k) = T_{i_k}$ for some element t_k of the set $\overline{T}_{j_{k-1}} \setminus T_{j_{k-1}}$, where $\overline{T}_{j_{k-1}}$ is the element of the chain D which covers the element $T_{j_{k-1}}$ in the chain D ,
i. e., $\overline{T}_{j_{k-1}} = T_{j_{k-1}+1}$;
- b) for any t from the set $T_{j_k} \setminus T_{j_{k-1}}$ which is different from t_k , the element $f_{j_k j_{k-1}}(t)$ is defined arbitrarily.

The set of all such mappings $f_{j_k j_{k-1}}$ of the set $T_{j_k} \setminus T_{j_{k-1}}$ in the set D'_k is denoted by $M_{j_k j_{k-1}}$. Next, in each set $M_{j_k j_{k-1}}$ ($k = 2, 3, \dots, s$) we choose a representative and denote the set of all chosen representatives of the set $M_{j_k j_{k-1}}$ by M .

Using the equality

$$\check{D} = T_{j_s} = T_{j_1} \bigcup_{k=2}^s (T_{j_k} \setminus T_{j_{k-1}}),$$

we can define, for $s \geq 2$, the mapping $f_{MT_{j_1}}$ of the set $\check{D} = T_{j_s}$ as follows:

$$f_{MT_{j_1}}(t) = \begin{cases} f_{j_k j_{k-1}} & \text{if } t \in T_{j_k} \setminus T_{j_{k-1}}, \\ T_{i_1} & \text{if } T_{j_1} \neq \emptyset, t \in T_{j_1}, \end{cases}$$

for any $t \in \check{D} = T_{j_s}$. Moreover, for $s = 1$, i. e., when $D' = \{T_{i_1}\}$ and $D'' = \{\check{D}\}$, the chains D of the mapping which correspond to these subchains are denoted by $f_{T_{i_1} \check{D}}$.

For all $t \in \check{D}$ they satisfy the condition $f_{T_{i_1} \check{D}}(t) = T_{i_1}$. In what follows such mappings $f_{MT_{j_1}}$ and $f_{T_{i_1} \check{D}}$ are called step mappings corresponding to the chains D'' and D' . The set of all such step mappings corresponding to the chains D'' and D' is denoted by $F_{D'' D'}$.

2000 *Mathematics Subject Classification*: 20M10, 20M20.

Key words and phrases. Semigroups of binary relations.

Theorem 1. Let D be a finite X -chain. Then for any regular binary relation α of the semigroup $B_X(D)$ there exist uniquely defined subchains D' and D'' of the chain D such that $D' \in C_0(s; D)$, $D'' \in C_1(s; D)$, a step mapping f corresponding to the chains D'' and D' , and a mapping f_1 of the set $X \setminus \check{D}$ in D' which satisfies the condition $\alpha = \alpha(f; f_1)$.

On the other hand, if D'_1 and D''_1 are such subchains of the chain D that $D'_1 \in C_0(n; D)$, $D''_1 \in C_1(n; D)$ and f' and f'_1 are respectively an arbitrary step mapping corresponding to the chains D''_1 and D'_1 and an arbitrary mapping of the set $X \setminus \check{D}$ in D'_1 , then the binary relation $\alpha' = \alpha'(f'; f'_1)$ is always a regular element of the semigroup $B_X(D)$.

Theorem 2. Let $D = \{T_0, T_1, \dots, T_m\}$ ($m \geq 1$) be an X -chain of the finite set X having a representation of the form $T_0 \subset T_1 \subset \dots \subset T_m$, where $k_{i,j} = |T_i \setminus T_j|$ ($0 \leq j \leq i \leq m$). Then the following statements are true:

If the element $D'' = \{T_{J_1}, T_{J_2}, \dots, T_{J_s}\}$ of the set $C_1(s; D)$ is represented as $T_{J_1} \subset T_{J_2} \subset \dots \subset T_{J_s}$ and $D' \in C_0(s; D)$, then the number $r(D''; s)$ of all regular elements of the semigroup $B_X(D)$, which corresponds to pairs of mappings $(f; f')$, where $f \in F_{D''D'}$ and f' is an arbitrary mapping of the set $X \setminus \check{D}$ in the s -element set D' , can be calculated by the formula

$$\begin{aligned} r(D''; s) = \\ = (2^{k_{J_1+1, J_1}} - 1) \cdot (3^{k_{J_2+1, J_2}} - 2^{k_{J_2+1, J_2}}) \dots (s^{k_{J_{s-1}+1, J_{s-1}}} - (s-1)^{k_{J_{s-1}+1, J_{s-1}}}) \times \\ \times 2^{k_{J_2 J_1+1}} \cdot 3^{k_{J_3 J_2+1}} \dots s^{k_{J_s J_{s-1}+1}} \cdot s^{|X \setminus \check{D}|}. \end{aligned}$$

If the set of all regular elements of the semigroup $B_X(D)$ is denoted by R_D , then

$$|R_D| = \begin{cases} |D| + \sum_{s=2}^m \left(C_m^s \cdot \sum_{D'' \in C_1(s; D)} r(D''; s) \right) & \text{if } \emptyset \notin D, \\ 1 + \sum_{s=2}^m \left(C_{m-1}^{s-1} \cdot \sum_{D'' \in C_1(s; D)} r(D''; s) \right) & \text{if } \emptyset \in D, \end{cases}$$

where $C_m^n = \frac{m!}{n!(m-n)!}$.

REFERENCES

1. Ya. I. Diasamidze, Complete semigroups of binary relations. (Russian) *Ajara, Batumi*, 2000.

Authors' address:

Batumi State University
35, Ninoshvili St., Batumi 384500
Georgia