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Semigroups $B_X(D)$ Defined by Finite X -Chains

Let X be an arbitrary nonempty set. Recall that a nonempty set D of subsets from X is called an X -chain if D is a chain with respect to a set-theoretic inclusion. In the sequel we consider those complete semigroups of binary relations $B_X(D)$ which are defined by finite X -chains (see [2]). Furthermore, by $\Sigma(D)$ we denote the set of all subsemilattices of the complete join X -semilattice D , any of whose elements D' satisfies the following three conditions:

- a) $\emptyset \in D'$ if $\emptyset \in D$;
- b) $\Lambda(D'; D'_t) \in D'$ for all $t \in \check{D}' = \cup D'$ (by $\Lambda(D'; D'_t)$ we denote the set-theoretic union of all lower boundaries of the set $D_t = \{Z \in D' \mid t \in Z\}$ in the semilattice D);
- c) $Z' = \bigcup_{t \in Z'} (\Lambda(D'; D'_t))$ for any nonempty Z' from D' .

The set $\Sigma(D)$ is called the defining set for regular elements of the semigroup $B_X(D)$.

Theorem 1. *Let $D = \{T_0, T_1, \dots, T_m\}$ ($m \geq 1$) and $D' = \{T_{i_1}, T_{i_2}, \dots, T_{i_s}\}$ ($1 \leq s \leq m$) be respectively a finite X -chain and a subchain of the chain D which can be respectively represented in the form*

$$T_0 \subset T_1 \subset \dots \subset T_m \text{ and } T_{i_1} \subset T_{i_2} \subset \dots \subset T_{i_s}.$$

If

$$\alpha = \bigcup_{k=1}^s (Y_k^\alpha \times T_{i_k})$$

is a normal representation of a binary relation α of the semigroup $B_X(D)$, then α is a regular element of this semigroup if and only if

$$Y_k^\alpha \cap T_{J_k} \neq \emptyset \text{ and } \bigcup_{p=1}^n Y_p^\alpha \supseteq T_{J_n},$$

where $k = 1, 2, \dots, s$ for $T_{J_1} \neq \emptyset$ and $k = 2, 3, \dots, s$ for $T_{J_1} = \emptyset$ and $n = 1, 2, \dots, s$ for some s -element subchain $D'' = \{T_{J_1}, T_{J_2}, \dots, T_{J_s}\}$ of the chain D which is represented in the form $T_{J_1} \subset T_{J_2} \subset \dots \subset T_{J_s}$.

Corollary 1. *Let $D = \{T_0, T_1, \dots, T_m\}$ ($m \geq 1$) and $D' = \{T_{i_1}, T_{i_2}, \dots, T_{i_s}\}$ ($1 \leq s \leq m$) be respectively a finite X -chain and a subchain of the chain D , which satisfy the conditions*

$$T_0 \subset T_1 \subset \dots \subset T_m \text{ and } T_{i_1} \subset T_{i_2} \subset \dots \subset T_{i_s}.$$

If a binary relation α represented in the normal form

$$\alpha = \bigcup_{k=1}^s (Y_k^\alpha \times T_{i_k})$$

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is a regular element of the semigroup $B_X(D)$, then for any s -element subchain $D'_1 = \{T_{q_1}, T_{q_2}, \dots, T_{q_s}\}$ of the chain D having a representation of the form $T_{q_1} \subset T_{q_2} \subset \dots \subset T_{q_s}$, the binary relation

$$\beta = \bigcup_{k=1}^s (Y_k^\alpha \times T_{q_k})$$

is also a regular element of the semigroup $B_X(D)$.

Theorem 2. Let $D = \{T_0, T_1, \dots, T_m\}$ ($m \geq 1$) be an X -chain of the finite set X such that $T_0 \subset T_1 \subset \dots \subset T_m$; $\Sigma(D)$ is a defining set for regular elements of the semigroup $B_X(D)$ (see [2]) and $k_{i,j} = |T_i \setminus T_j|$ ($0 \leq j \leq i \leq m$). If the chain $D' = \{T_{i_1}, T_{i_2}, \dots, T_{i_s}\}$ from $\Sigma'(D)$, where $\Sigma'(D) = \{D' \in \Sigma(D) \mid |D'| \geq 2\}$, satisfies the condition $T_{i_1} \subset T_{i_2} \subset \dots \subset T_{i_s}$, then for the number $s(D')$ of all right units of the semigroup $B_X(D')$ and the set of all idempotents I_D of the semigroup $B_X(D)$ the following statements are valid:

$$\begin{aligned} \text{a) } s(D') &= (2^{k_{i_2 i_1}} - 1) \cdot (3^{k_{i_3 i_2}} - 2^{k_{i_3 i_2}}) \cdot \dots \cdot (s^{k_{i_s i_{s-1}}} - (s-1)^{k_{i_s i_{s-1}}}) \cdot s^{|X \setminus T_{i_s}|}; \\ \text{b) } |I_D| &= \begin{cases} |D| + \sum_{D' \in \Sigma'(D)} s(D') & \text{if } \emptyset \notin D, \\ 1 + \sum_{D' \in \Sigma'(D)} s(D') & \text{if } \emptyset \in D \end{cases} \end{aligned}$$

(see [2]).

Corollary 2. Let $D = \{T_0, T_1, \dots, T_m\}$ ($m \geq 1$) be a finite X -chain having a representation of the form $T_0 \subset T_1 \subset \dots \subset T_m$; $D' = \{T_{i_1}, T_{i_2}, \dots, T_{i_s}\}$ and $D'' = \{T_{j_1}, T_{j_2}, \dots, T_{j_s}\}$ ($1 \leq s \leq m$) be the subchains of the chain D which satisfy the conditions

$$D' \in C_0(s; D), \quad D'' \in C_1(s; D), \quad T_0 \subset T_1 \subset \dots \subset T_m \quad \text{and} \quad T_{i_1} \subset T_{i_2} \subset \dots \subset T_{i_s}$$

(see [1]). Then a regular binary relation α of the semigroup $B_X(D)$ which has a normal representation of the form

$$\alpha = \bigcup_{k=1}^s (Y_k^\alpha \times T_{i_k})$$

and satisfies the conditions

$$Y_k^\alpha \cap T_{j_k} \neq \emptyset \quad \text{and} \quad \bigcup_{p=1}^n Y_p^\alpha \supseteq T_{j_n},$$

where $k = 1, 2, \dots, s$ for $T_{j_1} \neq \emptyset$ and $k = 2, 3, \dots, s$ for $T_{j_1} = \emptyset$ and $n = 1, 2, \dots, s$, is an idempotent element of this semigroup if and only if $T_{i_1} \subseteq T_{j_1}$ and $T_{j_{k-1}} \subset T_{i_k} \subseteq T_{j_k}$ for any k , where $k = 2, 3, \dots, s$.

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