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Extensions of Semimodules and the Takahashi Functor $\text{Ext}_\Lambda(C, A)$

Let Λ be a semiring with 1. A homomorphism $f : S \rightarrow S'$ of Λ -semimodules is called kernel-regular if $f(x) = f(y)$, $x, y \in S$ implies $x + k_1 = y + k_2$ for some $k_1, k_2 \in \text{Ker}(f)$ [1]. A Λ -semimodule C is called normal if there exist a projective Λ -semimodule P and a kernel-regular surjective Λ -homomorphism $\varepsilon : P \twoheadrightarrow C$ [2]. Suppose A is a Λ -semimodule and C is a normal Λ -semimodule. Let $\mu : R \hookrightarrow P$ denote the inclusion $R \hookrightarrow P$, where $R = \text{Ker}(\varepsilon)$. The Takahashi functor $\text{Ext}_\Lambda(C, A)$ is defined (see [2]) by

$$\text{Ext}_\Lambda(C, A) = \text{Coker} \left(\text{Hom}_\Lambda(\mu, A) : \text{Hom}_\Lambda(P, A) \rightarrow \text{Hom}_\Lambda(R, A) \right).$$

A short exact sequence $S \xrightarrow{\lambda} T \xrightarrow{\tau} Y$ of Λ -semimodules is an extension of S by Y (one would say “ Y by S ”) in Takahashi’s sense if the surjection τ is kernel-regular [3].

Let $\mathcal{E}xt_\Lambda \text{JT}(Y, X)$ be the category of sequences of the form $X \xleftarrow{f} S \xrightarrow{\lambda} T \xrightarrow{\tau} Y$ [4], where f is a Λ -homomorphism, and $S \xrightarrow{\lambda} T \xrightarrow{\tau} Y$ is a Takahashi extension. Define a set $\text{Ext}_\Lambda \text{JT}(Y, X)$ by

$$\text{Ext}_\Lambda \text{JT}(Y, X) = \mathcal{E}xt_\Lambda \text{JT}(Y, X) / \sim,$$

where $\sim$ is the smallest equivalence relation under which $X \leftarrow S \rightarrow T \rightarrow Y$ is equivalent to $X \leftarrow S' \rightarrow T' \rightarrow Y$ whenever there exists a commutative diagram of the form

$$\begin{array}{ccccccc} X & \xleftarrow{\quad} & S & \xrightarrow{\quad} & T & \xrightarrow{\quad} & Y \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ X & \xleftarrow{\quad} & S' & \xrightarrow{\quad} & T' & \xrightarrow{\quad} & Y \end{array}$$

In fact $\text{Ext}_\Lambda \text{JT}(Y, X)$ is an abelian monoid-valued bifunctor, contravariant in Y and covariant in X , and additive in both these variables (see [4]).

Theorem 1. *For every Λ -semimodule A and every normal Λ -semimodule C , there is a natural isomorphism*

$$\text{Ext}_\Lambda(C, A) \cong \text{Ext}_\Lambda \text{JT}(C, A)$$

of abelian monoids.

Let G be a Λ -module. A short exact sequence $G \xrightarrow{x} B \xrightarrow{\sigma} M$ of Λ -semimodules is an extension of G by M in Inassaridze’s sense if σ is kernel-regular and $x(g) + b = b$, $g \in G$, $b \in B$ implies $g = 0$ [5]. Let $\text{Ext}_\Lambda \text{I}(M, G)$ denote the set of congruence classes of Inassaridze extensions of G by M . (The extension $G \xrightarrow{x} B \xrightarrow{\sigma} M$ is congruent to the extension $G \xrightarrow{x'} B' \xrightarrow{\sigma'} M$ if there exists a Λ -isomorphism $\beta : B \rightarrow B'$ such

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that the diagram

$$\begin{array}{ccccc}
 G & \xrightarrow{x} & B & \xrightarrow{\sigma} & M \\
 \parallel & & \downarrow \beta & & \parallel \\
 G & \xrightarrow{x'} & B' & \xrightarrow{\sigma'} & M
 \end{array}$$

is commutative.)

Theorem 2. *Let A be a semiring (with 1) which is additively cancellative. For every A -module G and every normal A -semimodule C , there is a natural isomorphism*

$$\mathrm{Ext}_A(C, G) \cong \mathrm{Ext}_A I(C, G)$$

of abelian groups.

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