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Regular Elements of the Semigroups $B_X(D)$ determined by the Generalized Elementary X -Semilattices

In this paper, all regular elements of the semigroups $B_X(D)$ (see [1-2]) are described in the case when these semigroups are determined by generalized elementary X -semilattices.

Below are considered complete semigroups of binary relations $B_X(D)$ determined by complete semilattices $D = D' \cup \{Z_0, \check{D}\}$ which satisfy the following conditions:

- $|D'| \geq 2$, $\check{D} = \bigcup D'$ and Z_0 and $\check{D}$ are respectively the smallest and the largest elements of the set D ;
- the set D' , consisting of nonempty subsets of the set X , is an antichain;
- union of any two distinct elements of the set D' equals $\check{D}$.

It is easy to see that each complete join X -semilattice of the above class, has the fixed diagram depicted in Fig. 1.

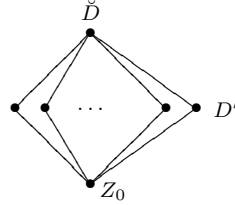


Fig. 1

Such semilattices will be in the sequel called generalized elementary semilattices.

Theorem 1. Let D be a generalized elementary X -semilattice and let $\alpha \in B_X(D)$. The binary relation α is a regular element of the semigroup $B_X(D)$ if and only if it satisfies at least one of the following conditions:

- $\alpha = X \times Z$ for some $Z \in D$;
- $\alpha = (Y \times Z) \cup ((X \setminus Y) \times Z')$ for some subset Y of X satisfying the conditions $Y \supseteq Z_0$ and $\check{D} \setminus Y \neq \emptyset$;
- $\alpha = (Y_0 \times Z_0) \cup (Y \times Z) \cup ((X \setminus (Y_0 \cup Y)) \times \check{D})$, for some decomposition Y_0, Y, Y_3 ($Y_3 = X \setminus (Y_0 \cup Y)$) of the set X satisfying the following conditions:

$$Y_0 \supseteq Z_0, \quad Y_0 \cup Y \supseteq T'', \quad T'' \setminus Y_0 \neq \emptyset, \quad \check{D} \setminus (Y_0 \cup Y) \neq \emptyset$$

for some T'' from D' .

- $\alpha = (Y_0 \times Z_0) \cup (Y \times Z) \cup ((X \setminus (Y_0 \cup Y)) \times Z')$ for some decomposition Y_0, Y, Y' ($Y' = X \setminus (Y_0 \cup Y)$) of the set X satisfying the following conditions:

$$Y_0 \supseteq T_1 \cap T_2, \quad T_1 \setminus Y_0 \neq \emptyset, \quad T_2 \setminus Y_0 \neq \emptyset, \quad Y_0 \cup Y \supseteq T_1,$$

$$Y_0 \cup Y' \supseteq T_2, \quad \check{D} \setminus (Y_0 \cup Y) \neq \emptyset, \quad Y_0 \cup Y' \supseteq \check{D}$$

for some distinct $T_1, T_2 \in D'$.

2000 Mathematics Subject Classification: 20M20, 20M10.

Key words and phrases. Semigroups of binary relations.

e) $\alpha = (Y_0 \times Z_0) \cup (Y \times Z) \cup (Y' \times Z') \cup ((X \setminus (Y_0 \cup Y \cup Y')) \times \check{D})$, for some decomposition Y_0, Y, Y', Y'' ($Y'' = X \setminus (Y_0 \cup Y \cup Y')$) of the set X satisfying the following conditions:

$$Y_0 \supseteq T_1 \cap T_2, \quad T_1 \setminus Y_0 \neq \emptyset, \quad T_2 \setminus Y_0 \neq \emptyset, \quad Y_0 \cup Y \supseteq T_1, \quad Y_0 \cup Y' \supseteq T_2,$$

$$\check{D} \setminus (Y_0 \cup Y) \neq \emptyset, \quad Y_0 \cup Y' \supseteq \check{D}, \quad Y_0 \cup Y \cup Y' \neq X$$

for some distinct $T_1, T_2 \in D'$.

Our further aim is to find formulæ for calculation of the number of regular elements of the semigroup $B_X(D)$,

Let X be a finite set, $D' \subseteq D$ and $D' = \{Z_1, Z_2, \dots, Z_n\}$.

A regular element of the semigroup $B_X(D)$ satisfying the condition t ($t \in \{a, b, c, d, e\}$) of theorem 1 will be called in the sequel an element of type t . The set of all regular elements of the semigroup $B_X(D)$ of type t will be denoted by R_t . Then the cardinality of the set R of all regular elements of the semigroup $B_X(D)$ can be calculated by the formula

$$|R| = |R_a| + |R_b| + |R_c| + |R_d| + |R_e|.$$

Theorem 2. For the finite set X , the number of all regular elements of type b) can be calculated by the formula

$$|R_b| = \begin{cases} (2n+1) \cdot (2^{|X \setminus Z_0|} - 2^{|X \setminus \check{D}|}), & \text{if } Z_0 \neq \emptyset, \\ (2n+1) \cdot (2^{|X \setminus Z_0|} - 2^{|X \setminus \check{D}|} - 1), & \text{if } Z_0 = \emptyset. \end{cases}$$

Theorem 3. For the finite set X , the number of all regular elements of type c) can be calculated by the formula

$$|R_c| = \begin{cases} n 3^{|X \setminus \check{D}|} \sum_{k=1}^n (2^{|Z_k \setminus Z_0|} - 1) (3^{|X \setminus Z_k|} - 2^{|X \setminus Z_k|}) & \text{if } Z_0 \neq \emptyset, \\ n 3^{|X \setminus \check{D}|} \left[\sum_{k=1}^n (2^{|Z_k|} - 1) (3^{|X \setminus Z_k|} - 2^{|X \setminus Z_k|}) - 2^{|X \setminus \check{D}|} \right] + n^2 2^{|X \setminus \check{D}|} & \text{if } Z_0 = \emptyset. \end{cases}$$

Theorem 4. For the finite set X , the number of all regular elements of type d) can be calculated by the formula

$$|R_d| = \frac{n(n-1)}{2} 3^{|X \setminus \check{D}|} \sum_{m,p=1}^n (2^{|Z_m \setminus Z_p|} - 1) (2^{|Z_p \setminus Z_m|} - 1).$$

Theorem 5. For the finite set X , the number of all regular elements of type e) can be calculated by the formula

$$|R_e| = \frac{n(n-1)}{2} (4^{|X \setminus \check{D}|} - 3^{|X \setminus \check{D}|}) \sum_{m,p=1}^n (2^{|Z_m \setminus Z_p|} - 1) (2^{|Z_p \setminus Z_m|} - 1).$$

Theorem 6. For the finite set X , the number $|R|$ of all regular elements of the semigroup $B_X(D)$ can be calculated by the following formulæ:

a) if $Z_0 \neq \emptyset$, then

$$\begin{aligned} |R| = & n + 2 + (2n+1) (2^{|X \setminus Z_0|} - 2^{|X \setminus \check{D}|}) + \\ & + n 3^{|X \setminus \check{D}|} \sum_{k=1}^n (2^{|Z_k \setminus Z_0|} - 1) (3^{|X \setminus Z_k|} - 2^{|X \setminus Z_k|}) + \\ & + \frac{n(n-1)}{2} 4^{|X \setminus \check{D}|} \sum_{m,p=1}^n (2^{|Z_m \setminus Z_p|} - 1) (2^{|Z_p \setminus Z_m|} - 1); \end{aligned}$$

b) if $Z_0 = \emptyset$, then

$$\begin{aligned} |R| &= 1 - n + (2n + 1)(2^{|X|} - 2^{|X \setminus \check{D}|}) + \\ &+ n3^{|X \setminus \check{D}|} \sum_{k=1}^n \left((2^{|Z_k|} - 1)(3^{|D \setminus Z_k|} - 2^{|D \setminus Z_k|}) - 2^{|X \setminus Z_k|} \right) \\ &+ n^2 2^{|X \setminus \check{D}|} + \frac{n(n-1)}{2} 4^{|X \setminus \check{D}|} \sum_{m,p=1}^n (2^{|Z_m \setminus Z_p|} - 1)(2^{|Z_p \setminus Z_m|} - 1). \end{aligned}$$

Example. Let $X = \{1, 2, 3\}$ and $D = \{\emptyset; \{2, 3\}; \{1, 3\}; \{1, 2\}; \{1, 2, 3\}\}$. Then

$$\begin{aligned} |R| &= 1 - 3 + 7(2^3 - 2^0) + 3 \cdot 3^0 \cdot 3((2^2 - 1)(3^1 - 2^1) - 2^1) + \\ &+ 9 \cdot 2^0 + 3 \cdot 4^0 \cdot 3(2^1 - 1)(2^1 - 1) = -2 + 49 + 9 + 9 + 9 = 74. \end{aligned}$$

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