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S-Minimal Non-nilpotent Lie Algebras

Let Σ be any abstract property. An universal algebra G with this property is called Σ -algebra and minimal non Σ -algebra if all proper subalgebras of G are Σ -algebras and G is not Σ -algebra. Classification of the classes of minimal non Σ -algebras for different Σ has long history (see monographs on universal algebras).

For minimal non-nilpotent Lie algebras see [1-3]. If L is a finite-dimensional non-nilpotent Lie algebra over an algebraically closed field all of whose proper subalgebras are nilpotent then L is the two-dimensional Lie algebra $L = \langle x, y \rangle$, $[x, y] = y$. Only observe here that there is an element x in L such that $\text{ad}(x)$ has a nonzero eigenvalue λ . Therefore there exists an element $y \in L$ with $[x, y] = y$. Since $\langle x, y \rangle$ is non-nilpotent we conclude $L = \langle x, y \rangle$.

So any two-dimensional nonabelian Lie algebra over a field F is minimal non-nilpotent. On the other hand if the similar Lie algebra is defined over the principal ideal domain K which is not a field and L is torsion free as a K -module then L is not minimal non-nilpotent. Subalgebra $(n)L = \langle nx, ny \rangle$, where $n \in K$ is not invertible is not nilpotent

$$[nx, ny] = n^2y.$$

This gives rise for the

Definition. A Lie algebra L is called *S*-minimal non Σ -algebra if L is not Σ -algebra and for every proper subalgebra $L_1 \subset L$ the isolator $I(L_1) = L$ or L_1 is Σ -algebra.

A subalgebra A of L is called isolated if for every $x \in L$, $A \cap \langle x \rangle = A$. For a subalgebra A the isolator $I(A)$ is the smallest isolated subalgebra contained A .

In this note we study infinite-dimensional solvable minimal non-nilpotent Lie algebras over any principal ideal domain.

Finite-dimensional solvable minimal non-nilpotent Lie algebras over arbitrary fields have been studied by E. Stitzinger [2] and A. Gein and S. Kuznecov [1]. In [3] D. W. Towers gave a fairly complete description of this algebras over great number of fields. In [4] M. F. Newman and I. Wiegold studied infinite minimal non-nilpotent p -groups. For up to date state of the topic see [5].

Theorem. *Let L be a torsion free infinite-dimensional solvable *S*-minimal non-nilpotent Lie algebra over a principal ideal domain K . Then the following conditions hold*

- (i) $L = L' \lambda \langle x \rangle$, where $L' = \sum_{i=1}^{\infty} \langle x_i \rangle$, $[x_1, x] = 0$, $[x_i, x] = x_{i-1}$, $i > 1$;
- (ii) L is hypercentral of length $\omega + 1$. The ideals of L are the terms of the ascending central series of L , i. e. the lattice of ideals of L is a chain;
- (iii) Every subalgebra of L is either a subideal of defect 2 or ascendent of type ω ;
- (iv) The set of subideals of L form a modular lattice.

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Remark. Dimension of a torsion free Lie algebra L over a ring K is the rank of the K -module L . If L is a Lie algebra which contains subalgebras A and B such that $L = A + B$, $A \cap B$ and A is an ideal of L , we say L splits over A and write $L = A\lambda B$.

Example. Of a minimal non-nilpotent algebra which is solvable of derived length 3. Consider $L = \langle x_1, x_2, \dots, x_i, \dots, x \rangle$ over a field of char 2 and with multiplications

$$\begin{aligned} [x_1, x] &= 0, \\ [x_i, x] &= x_{i-1}, \text{ for all } i > 1, \\ [x_i, x_j] &= 0 \text{ if } i + j = 2n \text{ is even,} \\ [x_i, x_j] &= z \text{ if } i + j = 2n + 1 \text{ is odd,} \\ [x_i, z] &= [x, z] = 0 \text{ for all } i \geq 1. \end{aligned}$$

1. L is not nilpotent: for every $n \in \mathbb{N}$ we have

$$0 \neq x_1 = [\dots [x_{n+1}, x], \dots, x];$$

$\underbrace{\hspace{1.5cm}}_{n\text{-times}}$

2. L is solvable of length 3:

$$\langle z \rangle \triangleleft \langle x_1, x_2, \dots, x_i, \dots, z \rangle \triangleleft L$$

is the derived series of L ;

3. L is minimal non-nilpotent: Let U be arbitrary proper subalgebra of L . Since $L/\langle z \rangle$ is minimal non-nilpotent we have $U + \langle z \rangle / \langle z \rangle \simeq U/U \cap \langle z \rangle$ is nilpotent which implies nilpotency of U .
4. But L is not Lie algebra $[[x_1, x_3], x] + [[x_3, x], x_1] + [[x, x_1], x_3] = z \neq 0$.
5. L is a binary Lie algebra, i. e. every two elements generate a Lie algebra.

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