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Perspective Maps over Rings

One of the main problems in the field of Geometric Algebra is to translate specific maps (perspectives, collineations, harmonic maps, etc.) from the geometric viewpoint into the algebraic language – which is the content of the Fundamental Theorems of the Geometric Algebra.

J. Dieudonné, R. Baer and J. von Neumann pointed out a possible extension of the structural identity between (projective) geometry and linear algebra to the case of a ring, generating intense research activity in the area of geometric algebra over rings [1], [2], [3]. Note that not a single ring version of the theorem about perspective maps is known.

A ring R will be called an SF-ring if each submodule of any free R -module is free. Let Ω be a free module over an SF-ring R , $\dim \Omega = N \geq 3$, $N < \infty$. Let M and M' be free submodules of the same rank $2 \leq n < N$.

Definition 1. A submodule $T \subset \Omega$ is called Δ -complement of the submodule M , if

$$M \cap T = 0, \quad M \oplus T \cong \Omega.$$

Proposition 1. The submodules M and M' admit a common Δ -complement T , i.e.,

$$M \cap T = M' \cap T = 0, \quad M \oplus T \cong M' \oplus T \cong \Omega.$$

Proposition 2. (I). $(M \oplus T) \cap (M' \oplus T) = \Omega_1 \cong \Omega$.

(II). If $M_1 = M \cap \Omega_1$, $M'_1 = M' \cap \Omega_1$ then $M_1 \oplus T = M'_1 \oplus T = \Omega_1$;

(III). If $U_1 \subset M_1$, $W \subset \Omega_1$, $T \subset W$, then $U_1 \subset M_1 \cap W \iff W = U_1 \oplus T$.

Let $U_1 \subset M_1$ be arbitrary submodule and let us form $W = U_1 \oplus T$. Consider also the intersection $W \cap M' = U'_1$. One has $W = U'_1 \oplus T$. Then between submodules $U_1 \subset M_1$ and submodules $U_2 \subset M_2$ there is a one-to-one correspondence, which is uniquely determined by the equality

$$U_1 \oplus T = U'_1 \oplus T.$$

Let $\mathbb{P}(M_1)$ and $\mathbb{P}(M'_1)$ be the projective spaces, i. e. the sets of rank 1 free submodules which are direct summands;

$$\mathbb{P}(M_1) = \{ \text{Re, where } e \in M_1 \text{ is a unimodular element} \}.$$

Definition 2. A bijection $p : \mathbb{P}(M_1) \rightarrow \mathbb{P}(M'_1)$ is called a perspective with the perspective center $\mathbb{P}(T)$ if it is determined by the equality

$$\text{Rm} \oplus T = \text{Rm}' \oplus T, \quad \text{Rm} \in \mathbb{P}(M_1), \quad \text{Rm}' \in \mathbb{P}(M').$$

Theorem 1. The perspective $p : \mathbb{P}(M_1) \rightarrow \mathbb{P}(M'_1)$ is induced by a linear map $\alpha : M_1 \rightarrow M'_1$ which fixes elements of the intersection $M_1 \cap M'_1$.

Definition 3. A bijection $\Delta p : \mathbb{P}(M) \rightarrow \mathbb{P}(M')$ is called a Δ -perspective with center $\mathbb{P}(T)$ if there exist submodules $M_1 \subset M$ and $M'_1 \subset M'$ such that between $\mathbb{P}(M_1)$ and

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$\mathbb{P}(M'_1)$ can be established a perspective p with center $\mathbb{P}(T)$ and the following diagram be commutative:

$$\begin{array}{ccc} \mathbb{P}(M_1) & \xrightarrow{p} & \mathbb{P}(M'_1) \\ \downarrow i & & \downarrow i' \\ \mathbb{P}(M) & \xrightarrow{\Delta p} & \mathbb{P}(M') \end{array} \quad \Delta p \circ i = i' \circ p,$$

$$i : Rm \rightarrow Re, \quad Rm = Re \cap M_1, \quad Rm \in \mathbb{P}(M_1), \quad Re \in \mathbb{P}(M);$$

$$i' : Rm' \rightarrow Re', \quad Rm = Re' \cap M'_1, \quad Rm' \in \mathbb{P}(M'_1), \quad Re' \in \mathbb{P}(M').$$

Theorem 1'. A Δ -perspective $\Delta p : \mathbb{P}(M) \rightarrow \mathbb{P}(M')$ is induced by a linear map $\alpha : M_1 \rightarrow M'_1$ which fixes elements of the intersection $M \cap M' = M_1 \cap M'_1$.

Lemma. A linear map $\alpha : M_1 \rightarrow M'_1$ fixes elements of the intersection $M_1 \cap M'_1$ if and only if the following representations hold:

$$M_1 = (M_1 \cap M'_1) \oplus M_0, \quad M'_1 = (M_1 \cap M'_1) \oplus M'_0.$$

A bijection $\beta : \mathbb{P}(M_1) \rightarrow \mathbb{P}(M'_1)$ is called a collineation if $Re_1 \subset Re_2 + Re_3 \iff \beta(Re_1) \subset \beta(Re_2) + \beta(Re_3)$; it is called a projection if it can be obtained as a result of successive application of perspectives.

Theorem 2. A collineation $\beta : \mathbb{P}(M_1) \rightarrow \mathbb{P}(M'_1)$ is a perspective if and only if it is induced by a linear map between the modules M_1 and M'_1 which fixes elements of the intersection $M_1 \cap M'_1$.

Theorem 3. Let $\beta : \mathbb{P}(M_1) \rightarrow \mathbb{P}(M'_1)$ be a collineation which fixes elements of the intersection

$$\mathbb{P}(M_1) \cap \mathbb{P}(M'_1) = \mathbb{P}(M_1 \cap M'_1).$$

Then

- 1) if $\dim(M_1 \cap M'_1) \geq 2$, then β is a perspective;
- 2) if $M_1 \cap M'_1 = 0$ and β is a projection, then β is a perspective;
- 3) if $\dim(M_1 \cap M'_1) = 1$, β is a projection and R is commutative, then β is a perspective.

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