

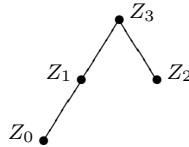
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**Semigroups $B_X(D)$ Determined by Semilattices
of Classes $\Sigma_3(X, 4)$ and $\Sigma_4(X, 4)$**

In this note, join X -semilattice classes $\Sigma_3(X, 4)$ and $\Sigma_4(X, 4)$ and semigroups $B_X(D)$ determined by these classes are considered (see [1,2]). Idempotent elements and maximal subgroups of the semigroups $B_X(D)$ are described. Formulæ for calculation of idempotent elements are given.

Let X be any non-empty set and Z_0, Z_1, Z_2, Z_3 be some subsets of the set X .

Below we will consider complete semigroups of binary relations $B_X(D)$ determined by such join X -semilattices $D = \{Z_0, Z_1, Z_2, Z_3\}$ whose elements satisfy the following conditions: $Z_2 \subset Z_3, Z_0 \subset Z_1 \subset Z_3, Z_0 \cup Z_2 = Z_3, Z_1 \cup Z_2 = Z_3, Z_0 \setminus Z_2 \neq \emptyset, Z_2 \setminus Z_0 \neq \emptyset$. Obviously the diagram of join X -semilattices having given properties is fixed and has the form given below:



In the given case, each binary relation α from the complete semigroup of binary relations $B_X(D)$ can be written in the following form:

$$\alpha = (Y_0^\alpha \times Z_0) \cup (Y_1^\alpha \times Z_1) \cup (Y_2^\alpha \times Z_2) \cup (Y_3^\alpha \times Z_3),$$

where Y_i^α ($i = \overline{0,3}$) are pairwise disjoint subsets of the set and $\bigcup_{i=0}^3 Y_i^\alpha = X$.

Lemma. Each join X -semilattice $D = \{Z_0, Z_1, Z_2, Z_3\}$ is uniquely determined by a set $\{C_2, P_{11}, P_{12}, P_{13}\}$ of pairwise disjoint subsets of the set X , where $|C_2| \geq 0, |P_{11}| \geq 1, |P_{12}| \geq 1, |P_{21}| \geq 1$ and $Z_0 = C_2 \cup P_{12}, Z_1 = C_2 \cup P_{12} \cup P_{21}, Z_2 = C_2 \cup P_{11} \cup P_{21}, Z_3 = C_2 \cup P_{11} \cup P_{12} \cup P_{21}$.

Corollary 1. 1) $Z_1 \cap Z_2 \neq \emptyset$.

2) $|Z_0| \geq 1, |Z_1| \geq 2, |Z_2| \geq 2, |Z_3| \geq 3$ and $|X| \geq 3$.

Note that if $C_2 \neq \emptyset$, then $Z_0 \cap Z_2 \neq \emptyset$. The class of join X -semilattices D corresponding to this case will be denoted by $\Sigma_3(X, 4)$, whereas if $C_2 = \emptyset$, then $Z_0 \cap Z_2 = \emptyset$ and the class of join X -semilattices D corresponding to this case will be denoted by $\Sigma_4(X, 4)$.

Corollary 2. For any semilattice D belonging to the class of semilattices $\Sigma_3(X, 4) \cup \Sigma_4(X, 4)$, the semigroup $B_X(D)$ does not possess right unit.

Corollary 3. For any semilattice D belonging to the class of semilattices $\Sigma_3(X, 4) \cup \Sigma_4(X, 4)$, every element of the set $B = \{\alpha \in B_X(D) | V(X^* \alpha) = D\}$ is an externally adjoined element to the semigroup $B_X(D)$.

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Theorem 1. Let $D \in \Sigma_3(X, 4)$, then a binary relation α is an idempotent element of the semigroup $B_X(D)$ if and only if α satisfies one of the following three conditions:

- 1) $\alpha = X \times Z'$ for some $Z' \in D$;
- 2) $\alpha = (Y^\alpha \times Z) \cup ((X \setminus Y^\alpha) \times Z')$ for some $Z, Z' \in D$, where $Z \subset Z'$ and Y^α is some non-empty subset of the set X satisfying the conditions $Y^\alpha \supseteq Z$ and $Z' \setminus Y^\alpha \neq \emptyset$;
- 3) $\alpha = (Y_0^\alpha \times Z_0) \cup (Y_1^\alpha \times Z_1) \cup ((X \setminus (Y_0^\alpha \cup Y_1^\alpha)) \times Z_3)$, where Y_0^α and Y_1^α are disjoint non-empty subsets of X satisfying the conditions $Y_1^\alpha \supseteq Z_0$, $Y_1^\alpha \cap Z_1 \neq \emptyset$, $Y_0^\alpha \cup Y_1^\alpha \supseteq Z_1$ and $Z_3 \setminus (Y_0^\alpha \cup Y_1^\alpha) \neq \emptyset$.

Theorem 2. Let $D \in \Sigma_4(X, 4)$, then a binary relation α is an idempotent element of the semigroup $B_X(D)$ if and only if α satisfies one of the conditions of theorem 1 or of the following two conditions:

- 4) $\alpha = (Y_0^\alpha \times Z_0) \cup ((X \setminus Y_0^\alpha) \times Z_2)$, where Y_0^α is some non-empty subset of X and $Z_0 \subseteq Y_0^\alpha \subseteq X \setminus Z_2$.
- 5) $\alpha = (Y_0^\alpha \times Z_0) \cup (Y_2^\alpha \times Z_2) \cup ((X \setminus (Y_0^\alpha \cup Y_2^\alpha)) \times Z_3)$, where Y_0^α and Y_2^α are disjoint non-empty subsets of X and $Z_0 \subseteq Y_0^\alpha \subseteq X \setminus Z_2$, $Z_2 \subseteq Y_2^\alpha \subseteq X \setminus Z_0$, $X \setminus (Y_0^\alpha \cup Y_2^\alpha) \neq \emptyset$.

Theorem 3. If $D \in \Sigma_3(X, 4)$, then the number of idempotent elements in the semigroup $B_X(D)$ is calculated by the formula

$$|I| = 4 + (2^{K_{32}} - 1)2^{|X \setminus Z_3|} + (2^{K_{30}} - 1)2^{|X \setminus Z_3|} + (2^{K_{31}} - 1)2^{|X \setminus Z_3|} + \\ + (2^{K_{10}} - 1)2^{|X \setminus Z_1|} + (2^{K_{10}} - 1)(3^{K_{31}} - 2^{K_{31}})3^{|X \setminus Z_3|}$$

whereas, if $D \in \Sigma_4(X, 4)$, then by the formula

$$|I'| = 4 + (2^{K_{32}} - 1)2^{|X \setminus Z_3|} + (2^{K_{30}} - 1)2^{|X \setminus Z_3|} + (2^{K_{31}} - 1)2^{|X \setminus Z_3|} + \\ + (2^{K_{10}} - 1)2^{|X \setminus Z_1|} + (2^{K_{10}} - 1)(3^{K_{31}} - 2^{K_{31}})3^{|X \setminus Z_3|} + 3^{|X \setminus Z_3|},$$

where $K_{ij} = |Z_i \setminus Z_j|$ ($i > j$).

Denote by $G_X(D, \varepsilon)$ the maximal subgroup of the semigroup $B_X(D)$ whose unit is the idempotent element ε .

Theorem 4. If $D \in \Sigma_3(X, 4)$, then every maximal subgroup $G_X(D, \varepsilon)$ of the semigroup $B_X(D)$ is a unit group, whereas if $D \in \Sigma_4(X, 4)$ then the order of every maximal subgroup of the semigroup $B_X(D)$ does not exceed 2, i. e. $|G_X(D, \varepsilon)| \leq 2$.

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