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$A(\infty)$ -Algebras and Hochschild Cohomology

1. HOMOTOPY G -ALGEBRAS

The cochain complex $C^*(X)$ of a topological space is not commutative but it's non-commutativity is measured by Steenrod's $\smile_1$ product:

$$d(a \smile_1 b) + da \smile_1 b + a \smile_1 db = a \smile b - b \smile a. \quad (1)$$

What is the good notion of dga with measured noncommutativity? Of course just $(A, d, \smile, \smile_1)$, i.e. a dga $(A, d, \smile)$ equipped additionally with $\smile_1$ satisfying only (1) is too pure for this.

Seems to be the suitable notion is the notion of *homotopy G -algebra*, introduced by Gerstenhaber and Voronov [2], which is a (1-reduced) dga $(A, d, \cdot)$ together with a sequence of multilinear maps $E_{1,k} : A \otimes A^{\otimes k} \rightarrow A$, $k = 1, 2, \dots$, satisfying certain coherence conditions.

Particularly the *leading* operation $E_{1,1}$ satisfies (1), so we denote $E_{1,1}(a, b) = a \smile_1 b$. Besides, $E_{1,1}$ satisfies so called "left" Hirsch formula

$$(a \cdot b) \smile_1 c = a \cdot (b \smile_1 c) + (a \smile_1 b) \cdot c, \quad (2)$$

which is satisfied particularly in $C^*(X)$. Besides

$$\begin{aligned} a \smile_1 (b \cdot c) - b \cdot (a \smile_1 c) - (a \smile_1 b) \cdot c = \\ = dE_{1,2}(a; b, c) + E_{1,2}(da; b, c) + E_{1,2}(a; db, c) + E_{1,2}(a; b, dc), \end{aligned} \quad (3)$$

so the symmetric expression ("right" Hirsch formula) is satisfied just up to chain homotopy and the suitable homotopy is $E_{1,2}$. One more condition is

$$a \smile_1 (b \smile_1 c) - (a \smile_1 b) \smile_1 c = E_{1,2}(a; b, c) + E_{1,2}(a; c, b). \quad (4)$$

Generally that coherence condition means that the multiplication $BA \otimes BA \rightarrow BA$ induced by $\{E_{1,k}\}$ turns the bar construction BA into a dg bialgebra.

Main examples of hga's are:

1. Cochain complex of a topological space $C^*(X)$, where the leading operation $E_{1,1} = \smile_1$ is classical Steenrod $\smile_1$ and the "tail" $E_{k>1,1}$ follows from the Baues's diagonal in the cobar construction $\Omega C_*(X)$, see [1];
2. Hochschild cochain complex $C^*(A, A)$ of an associative algebra A , where $E_{1,1}$ is again classical Gerstenhaber "circle" product and the tail $E_{k>1,1}$ is formed by the operations constructed in [3].
3. The cobar construction ΩH of a dg bialgebra H , where $E_{1,1}$ was defined by Adams and the tail $E_{k>1,1}$ was defined in [4].

Below we concentrate on the second example, so here is the formula for $E_{1,k}$ in $C^*(A, A)$ from [3]:

$$\begin{aligned} E_{1,i}(f; g_1, \dots, g_i)(a_1 \otimes \dots \otimes a_n) = \\ = \sum f(a_1 \otimes \dots \otimes a_{k_1} \otimes g_1(a_{k_1+1} \otimes \dots \otimes a_{k_1+n_1}) \otimes \dots \otimes \end{aligned}$$

2000 *Mathematics Subject Classification*: 16E40, 57T30, 18G55.

Key words and phrases. Bar construction, $A(\infty)$ -algebras, Hochschild cohomology.

$$\otimes a_{k_i} \otimes g_i(a_{k_i+1} \otimes \cdots \otimes a_{k_i+n_i}) \otimes \cdots \otimes a_n).$$

1. Deligne conjecture. In 1993 Deligne conjectured that the *little square operad* acts on $C^*(A, A)$. The conjecture was proved in [5] using the above operations. Namely, they have proved that little square operad is equivalent to the suboperad $F_2\chi$ of the surjection operad χ . This suboperad is generated by elements $(1, 2, 1, 3, 1, \text{dots}, 1, k-1, 1)$ and these elements correspond to the above described operations $E_{1,k}$ which act in $C^*(A, A)$.

2. Gerstenhaber algebras. The condition (4) implies that the commutator $[a, b] = a \smile_1 b - b \smile_1 a$ satisfies the Jacobi identity. Together with (1) this commutator implies on the $H(A)$ a Lie bracket of degree -1. Besides (2) and (3) imply that $[a, -] : H(A) \rightarrow H(A)$ is a derivation, so homology $H(A)$ of a hGa $(A, d, \cdot, \{E_{1,k}\})$ becomes a *Gerstenhaber algebra*.

This structure is generally nontrivial in the Hochschild cohomology $H^*(A, A)$ but it is certainly trivial on $H(C^*(X)) = H^*(X)$ since of the existence of $\smile_2$ product in $C^*(X)$.

3. Twisting cochains in the Hochschild complex. Let $(A, \mu : A \otimes A \rightarrow A)$ be a graded algebra. In this case the Hochschild complex is bigraded: $C^{p,q}(A, A) = Hom^q(A^{\otimes p}, A)$. In [3] we have introduced the notion of Hochschild twisting cochain, this is $m = m_3 + m_4 + \cdots$, $m_i \in C^{i,2-i}(A, A)$ satisfying the condition $dm = m \smile_1 m$. So in this definition participates just $E_{1,1} = \smile_1$. The higher operations $E_{1,k}$ participate in the notion of *perturbation of twisting cochains*: for a particular $g = g_2 + g_3 + \cdots$, $g_i \in C^{i,1-i}(A, A)$ the cochain $m' = m + dg + m \smile_1 g + \sum_k E_{1,k}(m'; g, \dots, g)$ (this is recurrent formula: m' can be solved from this equation componentwise) is twisting. Particularly for $g = g_n$ we have

$$m' = m_3 + m_4 + \cdots + m_n + (m_{n+1} + dg_n) + m'_{n+2} + m'_{n+3} + \cdots. \quad (5)$$

A twisting cochain m we call *degenerate* if it can be perturbed to $m' = 0$.

The justification of these notions is that *any* twisting cochain m determines on A a structure of $A(\infty)$ -algebra $(A, \{m_1 = 0, m_2 = \mu, m_3, m_4, \dots\})$: the condition $dm = m \smile_1 m$ coincides with Stasheff's condition defining $A(\infty)$ -algebras. Moreover, if g perturbs m to m' then the suitable $A(\infty)$ -algebras are *isomorphic*. To trivial twisting cochain $m = 0$ corresponds $(A, \{m_1 = 0, m_2 = \mu, m_3 = 0, m_4 = 0, \dots\})$, i.e., just graded algebra A itself. So to a degenerate twisting cochain m corresponds to *formal* $A(\infty)$ -algebra, that is which is isomorphic to the graded algebra A .

Formula (5) allows to introduce obstructions for such formality. Namely, the condition $dm = m \smile_1 m$ implies $dm_3 = 0$. If the class of m_3 in $H^{3,-1}(A, A)$ is zero, then $m_3 = dg_2$, and perturbing m by $g = g_2$ we obtain $m' = 0 + m'_4 + m'_5 + \cdots$. Now m'_4 becomes cocycle, and if its class is zero in $H^{4,-2}(A, A)$ we can kill it too, etc. Finally we obtain

Theorem. *If for a graded algebra A all hochschild cohomologies $H^{k,2-k}(A, A) = 0$ for $k \geq 3$ then A is intrinsically formal, that is each $A(\infty)$ -algebra structure on A with $m_1 = 0$, $m_2 = \mu$ is degenerate.*

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