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Notions of Fibrewise Discreteness

In late eighties Anders Kock [1] introduced and studied a condition on continuous maps $f : X \to Y$ between topological spaces and, more generally, on geometric morphisms between toposes, which tended to express the property of the map to have discrete inverse images of points but was of logical nature. One of many ways to describe this condition is that the induced map $f^{-1} : \mathscr{O}(Y) \to \mathscr{O}(X)$ between complete Heyting algebras of open subsets satisfies the condition called by Kock *regularity*:

$$\forall_{U\in\mathscr{O}(X)} \bigwedge_{V\in\mathscr{O}(Y)} (U \to f^{-1}V) \to f^{-1}V \;=\; U. \tag{Reg}$$

Kock proposed to view this condition as a non-classical relativization of a form of the basic rule of classical logic,

$$\forall_{\varphi}\ \neg\neg\varphi = \varphi.$$

About the same time, Peter Johnstone [2] and the author [3] considered this and some other conditions formulated in terms of fibrewise denseness and fibrewise closedness; for example, the condition (Reg) is equivalent to requiring that X is the only fibrewise dense subspace of X. Another condition which is easily seen to be strictly stronger than (Reg) is

$$\forall_{U\in\mathscr{O}(X)} \bigvee_{V\in\mathscr{O}(Y)} U \leftrightarrow f^{-1}V \;=\; X \tag{Clp}$$

which can be viewed as a non-classical relativization of the law of excluded middle

$$\forall_{\varphi}\ \varphi \vee \neg\varphi.$$

These two conditions have been compared and studied in [4]. In fibrewise terms, (Clp) implies that every subspace of X is fibrewise closed, which in turn implies (Reg). Thus it is natural to ask whether this intermediate condition is equivalent to either of the previous. We have proved

Theorem. *For a continuous map $f : X \to Y$, every subspace of X is fibrewise closed if and only if the condition*

$$\forall_{U\in\mathscr{O}(X)} \bigwedge_{V\in\mathscr{O}(Y)} (U \leftrightarrow f^{-1}V) \to f^{-1}V \;=\; U \tag{Clo}$$

holds. Moreover, none of the implications (Clp) $\Rightarrow$ (Clo) $\Rightarrow$ (Reg) *can be reversed.*

References

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