

G. JANELIDZE

Protomodular and Semiabelian Categories

This is a short survey on protomodular and semiabelian categories based on the papers [1], [5] and others. We also discuss the relationship with the old ideas of S. Mac Lane [6], with some old and recent developments in universal algebra (in connection with [3]), and with categorical notions of a semidirect product [2] and of a crossed module [4].

A category $\mathbf{C}$ is said to be *Bourn protomodular* (i.e., protomodular in the sense of D. Bourn [1]) if it admits pullbacks of split epis along arbitrary morphisms, and all such pullback functors are conservative (i.e., reflect isomorphisms). Here, the morphisms of split epis should agree with the splittings; however it turns out that omitting this requirement would not change the definition. Moreover, the split epis can be further replaced with *descent morphisms*, and hence with regular epis provided those are pullback stable. If $\mathbf{C}$ has an initial object then the conservativity condition becomes nothing but a suitably formulated *Split Short Five Lemma*.

A category $\mathbf{C}$ is said to be *semiabelian* (in the sense of [5]), if it is Barr exact, Bourn protomodular, pointed, and admits finite coproducts. As shown in [5], every semiabelian category admits all finite colimits, and the semiabelianness is essentially equivalent to the so-called old list of axioms used in categorical homological algebra, radical theory, and isomorphism and decomposition theorems. If $\mathbf{C}$ is semiabelian, then the abovementioned pullback functors are not just conservative, but monadic - which leads to a categorical definition of the *semidirect product* [2]. That is, in some sense semiabelian categories can be described as Barr exact categories with “good” semidirect products. Another motivation for the prefix “semi-” is the observation that $\mathbf{C}$ is abelian if and only if both $\mathbf{C}$ and its opposite category are semiabelian.

As shown in [3], a variety of universal algebras is a semiabelian category if and only if it has 0-ary terms $e_1, \dots, e_n$, binary terms $t_1, \dots, t_n$, and $(n+1)$ -ary term t satisfying the identities $t(x, t_1(x, y), \dots, t_n(x, y)) = y$ and $t_i(x, x) = e_i$ for each $i = 1, \dots, n$. Since every variety of universal algebras is Barr exact and has all small colimits, this also gives a characterization of the semiabelian varieties (by adding the requirement that $e_1 = \dots = e_n$ must be the unique constant). It follows that the semiabelian varieties were in fact known to universal algebraists: as recently discovered by D. Bourn, the terms above appear in various old papers of A. Ursini and others.

Using the categorical notion of semidirect product [2], the *internal notion of a crossed module* is defined [4]; the category of internal crossed modules in a semiabelian category $\mathbf{C}$ is equivalent to the category of internal categories in $\mathbf{C}$ [4]. This can be considered as a first step in the development of nonabelian homological and homotopical algebra in semiabelian categories.

Many other interesting and important results on protomodular and semiabelian categories were obtained in last years by D. Bourn, also in his joint work with M. Gran, and most recently with F. Borceux.

REFERENCES

1. D. Bourn, Normalization equivalence, kernel equivalence, and affine categories. *Lecture Notes in Math.* **1488**, Springer, Berlin(1991), 43–62.

2000 *Mathematics Subject Classification*: 18A20, 18E10, 18B99.

Key words and phrases. Semiabelian category, protomodular category, exact category, short five lemma.

2. D. Bourn and G. Janelidze, Protomodularity, descent, and semidirect products. *Theory Appl. Categ.* **4**(1998), 37–46.
3. D. Bourn and G. Janelidze, Characterization of protomodular varieties of universal algebras. *Preprint 16/2002, Dept. Math. Instituto Superior Técnico, Lisbon*, 2002.
4. G. Janelidze, Internal crossed modules. *Georgian Math. J.* **10**(2003), No. 1, 99–114.
5. G. Janelidze, L. Marki, and W. Tholen, Semiabelian categories. *J. Pure Appl. Algebra* **168**(2002), 367–386.
6. S. Mac Lane, Duality for groups. *Bull. Amer. Math. Soc.* **56**(1950), 485–516.

Author's address:

A. Razmadze Mathematical Institute
 Georgian Academy of Sciences
 1, Aleksidze St., Tbilisi 0193
 Georgia