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The Number of Equivalences on a Finite Set

Recurrent formulas for calculating the number of all equivalence binary relations on a finite set are well known (see [1]). In this paper, formulas are found, which make it possible to carry out a straightforward calculation of the number of all equivalence binary relations on a finite set. Moreover, concurrently, a formula is derived, by means of which it can be established in how many ways one can choose, in an n -element set, pairwise nonintersecting subsets whose number is equal to m ($m \leq n$). Note that the latter formula can later be used to calculate the number of idempotent regular and externally adjoined elements of the semigroup $B_X(D)$ (see [2]).

We denote by $\Sigma(K_n)$ the set of all possible partitions of an arbitrary n -element set ($n > 1$), and by $K_n = |\Sigma(K_n)|$ the number of elements of the set $\Sigma(K_n)$. Now we can define the set $\Sigma(K_n^m)$ as follows:

$$\Sigma(K_n^m) = \{H \in \Sigma(K_n) \mid |H| = m\} \quad (m \leq n).$$

The number of elements of the latter set is denoted by K_n^m , i. e., $K_n^m = |\Sigma(K_n^m)|$. It is obvious that $K_n = \sum_{i=1}^n K_i^n$.

The set $\Sigma_m(K_n^m)$ is defined as follows: if $H = \{M_1, M_2, \dots, M_m\} \in \Sigma(K_n^m)$, then $H_i = \{M_i, M_1, M_2, \dots, M_{i-1}, M_{i+1}, M_{i+2}, \dots, M_m\} \in \Sigma_m(K_n^m)$. It is obvious, that $|\Sigma_m(K_n^m)| = m \cdot K_n^m$.

Further, let G be an arbitrary n -element set and construct the set $G \setminus \{p\}$, where p is the fixed element of the set G . Consider, for the set G and $G \setminus \{p\}$, the following sets: $\Sigma(K_n^m)$, $\Sigma(K_{n-1}^{m-1})$ and $\Sigma(K_{n-1}^m)$. One can easily establish the one-to-one correspondence between the sets $\Sigma(K_n^m)$ and $\Sigma(K_{n-1}^{m-1}) \cup \Sigma_m(K_{n-1}^m)$ and therefore the equality

$$|\Sigma(K_n^m)| = |\Sigma_m(K_{n-1}^m)| + |\Sigma(K_{n-1}^{m-1})|$$

is valid, i. e., the following recurrent formula is fulfilled:

$$K_n^m = mK_{n-1}^m + K_{n-1}^{m-1}.$$

Lemma 1. *The formula*

$$K_n^m = \sum_{i=1}^m \frac{(-1)^{m+1}}{(i-1)!(m-i)!} \cdot i^{n-1}, \quad (0.1)$$

where $n = 1, 2, 3, \dots$ and $m \leq n$, is fulfilled for the value K_n^m .

Theorem 1. *The number of all equivalence binary relations K_n on an n -element set is calculated by the formula*

$$K_n = \sum_{i=1}^n \left(\sum_{j=i}^n \frac{(-1)^{i+j}}{(j-i)!} \right) \frac{i^{n-1}}{(i-1)!},$$

where $n = 1, 2, 3, \dots$ and $m \leq n$.

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Now assume that H is the set consisting of pairwise nonintersecting nonempty subsets of an arbitrary n -element set whose number is equal to m . Denote the set of all such H by $\Sigma(Q_n^m)$. Denote by Q_n^m the number of elements of the set $\Sigma(Q_n^m)$. The set $\Sigma(P_n^m)$ is defined as follows: $\Sigma(P_n^m) = \Sigma(Q_n^m) \setminus \Sigma(K_n^m)$, where P_n^m is the number of elements of the latter set. It clearly follows that $Q_n^m = K_n^m + P_n^m$. Let us find P_n^m . Indeed, let G be an arbitrary n -element set. For this set we consider the sets $\Sigma(Q_n^m)$, $\Sigma(P_n^m)$ and $\Sigma_{m+1}(K_n^{m+1})$. As above, here too we establish the one-to-one correspondence of the set $\Sigma(P_n^m)$ with the set $\Sigma_{m+1}(K_n^{m+1})$. Therefore

$$|\Sigma(P_n^m)| = |\Sigma_{m+1}(K_n^{m+1})| = (m+1) \cdot K_n^{m+1},$$

i.e., $P_n^m = (m+1) \cdot K_n^{m+1}$. Now from the equality $Q_n^m = K_n^m + P_n^m$ we obtain $Q_n^m = K_n^m + (m+1) \cdot K_n^{m+1}$ and, taking into account formula (1), we ascertain that the following equality is true:

$$\begin{aligned} Q_n^m &= \sum_{i=1}^m \frac{(-1)^{m+i}}{(i-1)!(m-i)!} \cdot i^{n-1} + (m+1) \sum_{i=1}^{m+1} \frac{(-1)^{m+i+1}}{(i-1)!(m+1-i)!} \cdot i^{n-1} = \\ &= \sum_{i=1}^{m+1} \frac{(-1)^{m+i+1}}{(i-1)!(m+1-i)!}. \end{aligned}$$

Thus the following statement is valid.

Theorem 2. *The formula*

$$Q_n^m = \sum_{i=1}^{m+1} \frac{(-1)^{m+i+1}}{(i-1)!(m+1-i)!} \cdot i^n,$$

where $n = 1, 2, 3 \dots$ and $m \leq n$, is fulfilled for the value Q_n^m .

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