Oleh Perehuda, Natalya Denysenko, Svitlana Kushnirenko

APPROXIMATION OF DELAY FUNCTIONAL-DIFFERENTIAL EQUATIONS BY DELAY-FREE EQUATIONS IN BANACH SPACES

Abstract. This paper develops a comprehensive approximation framework for functional-differential equations with delays in infinite-dimensional Banach spaces. Particular attention is devoted to the replacement of the original delayed system by an associated evolutionary system of delay-free differential equations. Such an approach not only simplifies the analytical treatment of delayed dynamics, but also provides a foundation for numerical implementations and further qualitative analysis of infinite-dimensional control systems. The proposed approach involves partitioning the delay interval into subintervals and constructing the corresponding system of equations that approximates the behavior of the original system. The main result of the study shows that, as the mesh size of the partition tends to zero, the distance between the solutions of the delayed equation and those of the delay-free system also tends to zero.

რეზიუმე. ნაშრომი განვითარებელია კომპლექსური აპროქსიმაციული მოდელი უსასრულო-განზომილებიან ბანახის სივრცეებში დაგვიანებული ფუნქციონალურ-დიფერენციალური განტოლებებისთვის. განსაკუთრებული ყურადღება ეთმობა საწყისი დაგვიანებული სისტემის ჩანაცვლებას დაგვიანების არმქონე დიფერენციალურ განტოლებათა შესაბამისი ევოლუციური სისტემით. ასეთი მიდგომა არა მხოლოდ ამარტივებს დაგვიანების დინამიკის ანალიტიკურ დამუშავებას, არამედ ქმნის საფუძველს რიცხვითი რეალიზაციისთვის და უსასრულო-განზომილებიანი მართვის სისტემების შემდგომი თვისებრივი ანალიზისთვის. შემოთავაზებული მიდგომა გულისხმობს დაგვიანების ინტერვალის ქვეინტერვალებად დაყოფას და განტოლებათა შესაბამისი სისტემის აგებას, რომელიც აპროქსიმაციას უკეთებს საწყისი სისტემის ქცევას. კვლევის მთავარი შედეგი აჩვენებს, რომ როდესაც დაყოფის ბადის ზომა ნულისკენ მიისწრაფვის, მანძილი დაგვიანებული განტოლების ამონახსნებსა და დაგვიანების არმქონე სისტემის ამონახსნებს შორის აგრეთვე ნულისკენ მიისწრაფვის.

1   Introduction

Delay differential equations play an important role in modeling the behavior of real-world processes whose evolution depends on past states. Numerous examples of such models can be found, for instance, in [2,11]. Since the initial data are given in the form of (continuous or integrable) functions, the phase space of such equations is infinite-dimensional, which significantly complicates their analysis. One possible approach, developed in [3, 4], is an approximation scheme that replaces the initial value problem for a system with delay by a Cauchy problem for a system of ordinary differential equations. As the dimension of such systems increases, their solutions converge to the solution of the original initial value problem with delay. This scheme is based on the idea of M. M. Krasovskii, involving the Taylor expansion of the solution of the delayed equation. A similar method was applied in [8] to the problem of optimal control for systems with infinite delay.

As for other approaches, we note the work [1]. Here, the semigroup generated by the linear part of the equation is approximated by a semigroup of finite-dimensional operators, which in turn leads to the replacement of the original functional-differential system by its finite-dimensional approximations.

The study of the behavior of solutions of functional-differential equations by reducing them to the behavior of solutions of ordinary equations is considered in the work [10]. There, a global attractor is constructed, consisting of solutions of a system of ordinary differential equations, to which all solutions of a functional-differential system of equations are attracted. A similar approach to the study of functional and stochastic differential equations was applied in [5, 9].

This work is structured into an introduction and three sections. The introduction outlines the problem statement and provides a review of the relevant literature.

Section 2 introduces the necessary concepts and presents the main results. Section 3 details the proofs of these main results. Section 4 illustrates the applicability of the obtained results with examples involving partial differential equations.

2   Preliminaries and main results

Let X be a Banach space with the norm $\| \cdot \|_X$. By $C_h = C([-h,0]; X)$ we denote the space of X-valued, continuous functions $\varphi : [-h, 0] \mapsto X$ with the norm

$$\|\varphi\|_C = \sup_{t \in [-h,0]} \|\varphi(t)\|,$$

where $h > 0$ is the delay interval.

Let $A : X \mapsto X$ be an unbounded, closed, linear operator and A be an infinitely small generator of a strongly continuous semigroup $S(t) = e^{tA}$, $t \geq 0$, in H.

Consider an infinite-dimensional functional differential equation of the form

$$\begin{cases} du(t) = \left(Au(t) + f\left(t, u(t), \int\limits_{-h}^{0} u(t + \theta)\, d\theta \right) \right) dt, & t \in [0, T], \\ u(t) = \varphi(t), & t \in [-h, 0]. \end{cases} \tag{2.1}$$

On mapping f, we impose the following conditions:

(A1) $f : [0, T] \times X \times X \mapsto X$ is continuous with respect to the set of variables;

(A2) (linear growth condition) there exists a constant $L > 0$ such that

$$\|f(t, u, v)\| \leq L\big(1 + \|u\| + \|v\|\big),$$

for all t, u, v from the domain of definition;

(A3) (Lipshitz condition)

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq L\big(\|u_1 - u_2\| + \|v_1 - v_2\|\big),$$

for $t \in [0, T]$ and arbitrary t, u_1, u_2, v_1, v_2 from the domain of definition;

(A4) initial function $\varphi : [-h, 0] \mapsto X$ is continuous.

The solution to the initial problem (2.1) will be understood in a mild sense.

Definition 2.1. A function $u(t) \in X$ is called a mild solution of the initial value problem (2.1) on $[0, T]$ if:

(1) $u(t) = \varphi(t)$, $t \in [-h, 0]$;

(2) $u \in C([0, T], X)$;

(3) $u(t)$ satisfies the integral equation

$$u(t) = S(0)\varphi(0) + \int\limits_0^t S(t-s) f\left(s, u(s), \int\limits_{-h}^0 u(s+\theta)\, d\theta\right) ds. \tag{2.2}$$

It follows from [11] that if conditions (A1)–(A4) are met, the initial problem (2.1) has a unique soft solution on $[0, T]$ that satisfies the inequality

$$\sup_{t \in [0, T]} \|u(t)\|_X < \infty.$$

We construct the following system of evolutionary equations without delays, which we will call approximating, using equation (2.1). Let us fix $m \in N$ and divide the interval $[-h, 0]$ by the points $-\frac{hj}{m}$, $j = \overline{0, m}$ into m parts.

We define the functions $z_j(t) \in X$ as solutions to the following Cauchy problems:

$$dz_0 = \left(Az_0 + f\left(t, z_0(t), \frac{h}{m} \sum_{j=1}^m z_j(t)\right)\right) dt,$$

$$dz_j(t) = \frac{m}{h}\left(z_{j-1}(t) - z_j(t)\right), \quad t \in [0, T], \tag{2.3}$$

$$z_j(0) = \varphi\left(-\frac{hj}{m}\right), \quad j = \overline{0, m}.$$

Here, $z_0(t)$, the solution of the first equation, is understood in the mild sense, while the remaining m equations are considered in the classical sense. The derivative $\frac{dz_j(t)}{dt}$ is considered as a strong derivative with respect to the norm of the space X. From [6] it follows that the Cauchy problem (2.3) has a unique solution on $[0, T]$, where $z_0(t)$ satisfies (2.3) in the mild sense, while $z_j(t)$ satisfy the remaining m equations in the classical sense.

Definition 2.2. System (2.3) is called an approximating system for (2.1) if

$$\sup_{t \in [0, T]} \left\|u\left(t - \frac{h}{m} j\right) - z_j(t)\right\| \longmapsto 0, \ \ m \mapsto \infty, \ \ j = \overline{0, m}.$$

The main result is the following

Theorem 2.1. *Under assumptions* (A1)–(A4), *system* (2.3) *is an approximating system for the initial value problem* (2.1), *uniformly with respect to* $j = \overline{0, m}$, *that is,*

$$\sup_{j=\overline{0,m}} \sup_{t \in [0, T]} \left\|u\left(t - \frac{h}{m} j\right) - z_j(t)\right\| \longmapsto 0, \ \ m \mapsto \infty. \tag{2.4}$$

In what follows, we will need a lemma on the modulus of continuity.

Lemma 2.1. *Under assumptions* (A1)–(A4), *for the solution of the initial value problem, the following inequality holds:*

$$\sup_{|t_1 - t_2| \le l, \ t_1, t_2 \in [-h, T]} \|u(t_2) - u(t_1)\| \le C\big(T, \|\varphi\|_C, h, l\big) \longmapsto 0, \ \ l \mapsto 0. \tag{2.5}$$

The proof follows directly from the uniform continuity of $u(t)$ on $[-h, T]$.

3   Proof of main theorem

Proof of Theorem 2.1. Recall that $u(t)$ is continuous in X. Let us introduce the following smoothed function $u_\mu(t)$, constructed for each small $\mu > 0$ as follows:

$$u_\mu(t) = \frac{1}{\mu} \int\limits_{t}^{t+\mu} u(s)\, ds, \ \ t \in [-h, T]. \tag{3.1}$$

Moreover, for $t \geq T$, the function $u(s)$ is extended as a constant value by continuity.

From the properties of the Bochner integral, it follows that the function $u_\mu(t)$ is strongly smooth and

$$\dot{u}_\mu(t) = \frac{1}{\mu} \left(u(t + \mu) - u(t) \right). \tag{3.2}$$

Let us estimate the difference between $u(t)$ and $u_\mu(t)$,

$$\sup_{t \in [-h,T]} \| u(t) - u_\mu(t) \| = \frac{1}{\mu} \sup_{t \in [-h,T]} \left\| \int\limits_{t}^{t+\mu} (u(t) - u(s))\, ds \right\|$$

$$\leq \frac{1}{\mu} \int\limits_{t}^{t+\mu} \sup_{t,s} \| u(t) - u(s) \|\, ds \leq C_3\big(T, \|\varphi\|_C, \mu\big) \longrightarrow 0, \ \ \mu \to 0, \tag{3.3}$$

as a consequent of the lemma on the modulus of continuity.

Since all equations in system (2.3) starting from the second one are linear equations with bounded constant operators, they admit a unique strong solution.

We split system (2.3) into two subsystems and represent the solution in the form $z_j(t) = z_j^{(1)}(t) + z_j^{(2)}(t)$, where $z_j^{(1)}(t)$ is the solution of the system

$$\begin{cases} \dfrac{h}{m}\, \dot{z}_1^{(1)} = u(t) - z_1^{(1)}, \\[2mm] \dfrac{h}{m}\, \dot{z}_j^{(1)} = z_{j-1}^{(1)} - z_j^{(1)}, & j = \overline{1, m}, \\[2mm] z_j^{(1)}(0) = u\left(-\dfrac{hj}{m} \right), \end{cases} \tag{3.4}$$

and $z_j^{(2)}(t)$ is the solution of the system

$$\begin{cases} \dfrac{h}{m}\, \dot{z}_1^{(2)} = -z_1^{(2)} + (z_0 - u(t)), \\[2mm] \dfrac{h}{m}\, \dot{z}_j^{(2)} = z_{j-1}^{(2)} - z_j^{(2)}, & j = \overline{1, m}. \\[2mm] z_j^{(2)}(0) = 0, \end{cases} \tag{3.5}$$

Then

$$\sup_{t \in [0,T]} \left\| u\left(t - \frac{hj}{m} \right) - z_j(t) \right\| \leq \sup_{t \in [0,T]} \left\| u\left(t - \frac{hj}{m} \right) - z_j^{(1)}(t) \right\| + \sup_{t \in [0,T]} \| z_j^{(2)}(t) \|. \tag{3.6}$$

Let us denote $y_j(t) = u(t - \frac{hj}{m})$ and define the differences

$$N_j(t) = \| y_j(t) - z_1^{(1)}(t) \|, \ \ j = \overline{0, m}.$$

We now estimate the first term in (3.6). To this end, $u(t)$ can be expressed in the form

$$u(t) = u_\mu(t) - (u(t) - u_\mu(t)) = u_\mu(t) - u_1(t),$$

and $y_j(t) = y_j^{(1)} + y_j^{(2)}(t)$. Here,

$$y_j^{(1)}(t) = u_\mu\left(t - \frac{hj}{m}\right), \quad y_j^{(2)}(t) = u_1\left(t - \frac{hj}{m}\right).$$

As a result, system (3.4) is reduced to two subsystems, and $z_j^{(1)}(t) = u_j(t) + v_j(t)$. Here, $u_j(t)$ is a solution of the system

$$\begin{cases} \dfrac{h}{m}\,\dot{u}_1 = -u_1 + u_\mu, \\[2mm] \dfrac{h}{m}\,\dot{u}_j = -u_{j-1} - u_j, & j = \overline{2, m}, \\[2mm] u_j(0) = y_j^{(1)}(0) = u_\mu\left(-\dfrac{hj}{m}\right), \end{cases} \tag{3.7}$$

and $v_j(t)$ is a solution of the system

$$\begin{cases} \dfrac{h}{m}\,\dot{v}_1 = -v_1 + u_1, \\[2mm] \dfrac{h}{m}\,\dot{v}_j = v_{j-1} - v_j, & j = \overline{2, m}. \\[2mm] v_j(0) = y_j^{(2)}(0) = u_1\left(-\dfrac{hj}{m}\right) - u_\mu\left(-\dfrac{hj}{m}\right), \end{cases} \tag{3.8}$$

Then

$$\sup_{t\in[0,T]} \|y_j(t) - z_j^{(1)}(t)\| \leq \sup_{t\in[0,T]} \left\|y_j^{(1)}(t) + y_j^{(2)}(t) - u_j(t) - v_j(t)\right\|$$

$$\leq \sup_{t\in[0,T]} \|y_j^{(1)}(t) - u_j(t)\| + \sup_{t\in[0,T]} \|y_j^{(2)}(t)\| + \sup_{t\in[0,T]} \|v_j(t)\|. \tag{3.9}$$

We estimate each term in (3.9). Let us consider the first term of this inequality. We will make estimates when $j = 1$. Denote $\varepsilon_1(t) = u_1(t) - y_1^{(1)}(t)$ at that $\varepsilon_1(0) = 0$.

Then

$$\dot{\varepsilon}_1 = \dot{u}_1 - \dot{y}_1^{(1)} = \frac{m}{h}(u_\mu - u_1) - \dot{y}_1^{(1)} = \frac{m}{h}\left(-u_1 + \dot{y}_1^{(1)}\right) + \frac{m}{h}\left(u_\mu - \dot{y}_1^{(1)}\right) - \dot{y}_1^{(1)}$$

$$= -\frac{m}{h}\,\varepsilon_1 + \frac{m}{h}\left(u_\mu - \dot{y}_1^{(1)}\right) - y_1^{(1)} = -\frac{m}{h}\,\varepsilon_1 + \psi(t).$$

Thus, for $\varepsilon_1(t)$, we get a system of equations

$$\begin{cases} \dot{\varepsilon}_1 = -\dfrac{m}{h}\,\varepsilon_1 + \psi(t), \\[2mm] \varepsilon_1(0) = 0. \end{cases} \tag{3.10}$$

To estimate $\psi(t)$, we have

$$\psi(t) = \frac{m}{h}\left(u_\mu(t) - y_1^{(1)}(t)\right) - \dot{y}_1^{(1)}(t)$$

$$= \frac{m}{h}\left(u_\mu(t) - u_\mu\left(t - \frac{h}{m}\right)\right) - \dot{u}_\mu\left(t - \frac{h}{m}\right) = \frac{m}{h}\left(u_\mu(t) - u_\mu\left(t - \frac{h}{m}\right)\right) - \dot{u}_\mu\left(t - \frac{h}{m}\right)\frac{h}{m}.$$

It follows from (3.2) that $u_\mu(t)$ is strongly continuously differentiable. Therefore,

$$\|\psi(t)\| \leq \sup_{\tau\in[0,1]} \left\|\dot{u}_\mu\left(t - \frac{h}{m} + \tau\frac{h}{m}\right) - \dot{u}_\mu\left(t - \frac{h}{m}\right)\right\|$$

$$= \frac{1}{\mu}\left\|u\left(t - \frac{h}{m} + \mu + \tau\frac{h}{m}\right) - u\left(t - \frac{h}{m} + \tau\frac{h}{m}\right) - u\left(t - \frac{h}{m} + \mu\right) + u\left(t - \frac{h}{m}\right)\right\|$$

$$\leq \frac{1}{\mu} \left(\sup_{\tau \in [0,1]} \left\| u\left(t - \frac{h}{m} + \mu + \tau \frac{h}{m}\right) - u\left(t - \frac{h}{m} + \mu\right) \right\| \right.$$
$$\left. + \sup_{\tau \in [0,1]} \left\| u\left(t - \frac{h}{m} + \tau \frac{h}{m}\right) - u\left(t - \frac{h}{m}\right) \right\| \right).$$

So,

$$\sup_{t \in [0,T]} \|\psi(t)\| \leq \frac{1}{\mu} \sup_{t \in [0,T]} \sup_{\tau \in [0,1]} \left\| u\left(t - \frac{h}{m} + \mu + \tau \frac{h}{m}\right) - u\left(t - \frac{h}{m} + \mu\right) \right\|$$
$$+ \frac{1}{\mu} \sup_{t \in [0,T]} \sup_{\tau \in [0,1]} \left\| u\left(t - \frac{h}{m} + \tau \frac{h}{m}\right) - u\left(t - \frac{h}{m}\right) \right\| \leq C\left(T, \|\varphi\|_C, \frac{h}{m}\right) \longrightarrow 0, \quad m \to \infty,$$

due to the lemma about the modulus of continuity. With (3.10) and the formula for the variation of an arbitrary constant, we have

$$\varepsilon_1(t) = \int_0^t e^{-\frac{m}{h}(t-s)} \psi(s)\, ds.$$

Applying Bunyakovsky's inequality, we get

$$\sup_{t \in [0,T]} \left\| \int_0^t e^{-\frac{m}{h}(t-s)} \psi(s)\, ds \right\| \leq \sup_{t \in [0,T]} \left(\int_0^t e^{-\frac{m}{h}(t-s)}\, ds \int_0^t e^{-\frac{m}{h}(t-s)} \|\psi(s)\|\, ds \right)$$
$$\leq \frac{h}{m} \sup_{t \in [0,T]} \int_0^t e^{-\frac{m}{h}(t-s)}\, ds \cdot \sup_{t \in [0,T]} \leq \frac{h}{m} \frac{2}{\mu} C\left(T, \|\varphi\|, \frac{h}{m}\right). \quad (3.11)$$

Thus, for $\varepsilon_1(t)$, the following inequality holds:

$$\sup_{t \in [0,T]} \|\varepsilon_1(t)\| \leq \frac{2h}{m\mu} C\left(T, \|\varphi\|, \frac{h}{m}\right). \quad (3.12)$$

Next, we will estimate $y_1^{(2)}(t) = u_\mu(t - \frac{h}{m}) - u(t - \frac{h}{m})$. With (3.3), we have

$$\sup_{t \in [0,T]} \|y_1^{(2)}(t)\| = \sup_{t \in [0,T]} \left\| u_\mu\left(t - \frac{h}{m}\right) - u\left(t - \frac{h}{m}\right) \right\| \leq C\left(T, \|\varphi\|, \frac{h}{m}\right).$$

Now, let us estimate $v_1(t)$. From the variation formula for a linear equation, we have

$$v_1(t) = e^{-\frac{m}{h}t} v_1(0) + \int_0^t e^{-\frac{m}{h}(t-s)} u_1(s)\, ds.$$

Then

$$\sup_{t \in [0,T]} \|v_1(t)\| \leq \|v_1(0)\| + \sup_{t \in [0,T]} \left\| \int_0^t e^{-\frac{m}{h}(t-s)} u_1(s)\, ds \right\| \cdot \frac{h}{m}. \quad (3.13)$$

But $v_1(0) = u_1(-\frac{h}{m})$. Then $\|v_1(0)\| \leq C(T, \|\varphi\|, \mu)$, and therefore,

$$\sup_{t \in [0,T]} \left\| \int_0^t e^{-\frac{m}{h}(t-s)} u_1(s)\, ds \right\| \cdot \frac{h}{m} \leq \frac{h}{m} C(T, \|\varphi\|, \mu). \quad (3.14)$$

So, with (3.13) and (3.14), we have

$$\sup_{t \in [0,T]} \|v_1(t)\| \leq 2C(T, \|\varphi\|, \mu). \quad (3.15)$$

Then with (3.9), (3.12) and (3.15), we get

$$\sup_{t\in[0,T]} \|y_1(t) - z_1^{(1)}(t)\| \leq \frac{2h}{m\mu} C\Big(T, \|\varphi\|, \frac{h}{m}\Big) + 2C(T, \|\varphi\|, \mu).$$

After making a similar estimate for $j = 2$, we obtain the inequality

$$\sup_{t\in[0,T]} \|y_j(t) - z_j^{(1)}(t)\|$$

$$\leq 2hC\Big(T, \|\varphi\|, \frac{h}{m}\Big) + 3C\Big(T, \|\varphi\|, C^{\frac{1}{2}}\Big(T, \|\varphi\|, \frac{h}{m}\Big)\Big) = \alpha\Big(T, \|\varphi\|, \frac{h}{m}\Big) \longrightarrow 0, \ \ m \to \infty.$$

The corresponding estimates for system (3.5) take the form

$$\|z_j^{(2)}(t)\| = \sup_{t\in[0,T]} N_0(t).$$

Therefore, we get the inequality

$$\sup_{t\in[0,T]} N_j(t) \leq \alpha\Big(T, \|\varphi\|, \frac{h}{m}\Big) + \sup_{t\in[0,T]} N_0(t). \tag{3.16}$$

Let us estimate $N_0(t)$. We have

$$z_0(t) - u(t) = \int_0^t S(t-s)\Big(f(s, z_0(s), z_m(s)) - f\Big(s, u(s) \int_0^t u(s+\theta)\Big) d\theta\Big) ds.$$

Then

$$N_0(t) \leq MTL \int_0^t \Big(\|z_0(s) - u(s)\| + \|z_m(s)\| - \int_{-h}^0 \|u(s+\theta)\| d\theta \Big) ds$$

$$\leq 2MTL \int_{-h}^0 \sup_{\tau\in[0,s]} N_0(\tau)\, d\tau + \alpha\Big(T, \|\varphi\|, \frac{h}{m}\Big).$$

Applying Gronwall's lemmas, we arrive at

$$\sup_{t\in[0,T]} N_0(t) \leq 2\alpha\Big(T, \|\varphi\|, \frac{h}{m}\Big) e^{(T+1)} LMT \longrightarrow 0, \ \ m \to \infty.$$

The latest estimate, taking into account (3.16), proves the theorem. $\qquad\square$

4   Example

Let Q be a bounded domain in R^d with a bound of ∂Q satisfying the Lyapunov condition. The operator A is a second-order differential operator of the elliptic type

$$Au = \sum_{i,j=1}^d (a_{ij}(x)u_{x_i})_{x_j} = \mathrm{div}(a(x)\nabla u)).$$

Here, a_{ij} are Hölder-continuous coefficients with the Hölder exponent $\beta \in (0, 1)$, symmetric, bounded, and satisfying the condition of uniform ellipticity

$$\sum_{i,j=1}^d a_{ij}\eta_i\eta_j \geq C_0|\eta|^2, \ \ \eta \in R^d,$$

for some constant $C_0 > 0$, and $|\cdot|$ is the Euclidean norm in R^d.

Let $X = L^2(Q)$, $D(A) = H^2(Q) \cap H$.

Consider the following equation:

$$
\begin{cases}
du(t,x) = \left(Au + f\left(t, u(t), \int\limits_{-h}^{0} u(t+\theta)\,d\theta \right) \right) dt, \\[2mm]
u(t,x) = \varphi(t,x), \quad t \in [-h,0], \\[1mm]
u(0,x) = \varphi_0(x), \quad x \in Q, \\[1mm]
u(t,x) = 0, \quad x \in \partial Q, \quad t \geq 0,
\end{cases}
\tag{4.1}
$$

where $\varphi(t,x) \in C_0 = C([0,T]; L^2(Q))$.

The real-valued function $f(t,x,y)$ is defined for $t \in [0,T]$, $x \in Q$, $y \in [0,l]$, $l > 0$, with a value in R^1. The function $f(t,x,y)$ is continuous over the set of variables and Lipschitz over the variables x and y with a constant L.

The domain $D \in [-h,T] \times C$ is the set $\{(t,\varphi) : \ t \in [-h,T], \ \varphi \in G\}$, where G is the set of functions $\varphi \in C$ such that $\int\limits_{-h}^{0} \varphi(\theta, \cdot)\,d\theta \in (0,l)$, and ∂G is the set of functions $\varphi \in C$ such that $\int\limits_{-h}^{0} \varphi(\theta, \cdot)\,d\theta = l$ or $\varphi(\theta,x) = 0$ a.e.

It follows from [6] that the operator A is a generator of C_0, a semigroup of operators $S(t) : X \mapsto X$. It is not difficult to see that the conditions (A1)–(A4) for equation (4.1) are fulfilled.

Indeed, for condition (A2), we have

$$
\|f(t,u,v)\|_{L^2(Q)}^2 = \int\limits_{Q} \left| f(t,u(x),v(x)) \right|^2 dx \leq L^2 \int\limits_{Q} \left(1 + |u(x)| + |v(x)| \right)^2 dx
$$

$$
\leq C \left(1 + \int\limits_{Q} |u(x)|^2\,dx + \int\limits_{Q} |v(x)|^2\,dx \right) = C\left(1 + \|u(x)\|^2 + \|v(x)\|^2 \right).
$$

Condition (A3) is checked similarly.

So, for equation (4.1), the statement of the theorem is valid.

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Authors' addresses:

Oleh Perehuda

Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska Str., Kyiv 01601, Ukraine
E-mail: perehudao@knu.ua

Natalya Denysenko

"I. Sikorsky Kyiv Polytechnic Institute" National Technical University of Ukraine, 37 Peremohy
Ave., Kyiv 03056, Ukraine
E-mail: nldenisenko@gmail.com

Svitlana Kushnirenko

Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska Str., Kyiv, 01601, Ukraine
E-mail: svilana.kushnirenko@knu.ua