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ON SOME BOUNDARY VALUE PROBLEMS CONNECTED WITH THE WHITHEY PROBLEM

Abstract. The existence of maximal and minimal solutions to some boundary value problems in the segment given by well-ordered lower and upper functions is proved. The general results obtained are applied to three modifications of the Whitney problem.

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1 Statement of the problem

On the interval $[a,b]$, we consider the problem

$$u'' = f(t,u,u'), \tag{1.1}$$

$$u'(a) = c, \quad u(a) = u(b), \tag{1.2}$$

where $f \in Car([a,b] \times \mathbb{R}^2; \mathbb{R})$ and $c \in \mathbb{R}$. As usual, by a solution to equation (1.1) we understand a function $u \in AC^1([a,b];\mathbb{R})$ satisfying the given equation almost everywhere on $[a,b]$. A solution u to equation (1.1) satisfying conditions (1.2) is said to be a solution to problem (1.1), (1.2).

Definition 1.1. We say that $\alpha \in AC_\ell([a,b];\mathbb{R})$ is a lower function and $\beta \in AC_u([a,b];\mathbb{R})$ is an upper function of equation (1.1) if $\alpha''(t) \geq f(t,\alpha(t),\alpha'(t))$, $\beta''(t) \leq f(t,\beta(t),\beta'(t))$ for a.e. $t \in [a,b]$. The lower and upper functions α and β of equation (1.1) are said to be well ordered if

$$\alpha(t) \leq \beta(t) \ \text{ for } \ t \in [a,b]. \tag{1.3}$$

We also consider the following standard two-point boundary conditions:

$$u(a) = c_1, \quad u(b) = c_2, \tag{1.4}$$

$$u'(a) = c_1, \quad u(b) = c_2, \tag{1.5}$$

where $c_1, c_2 \in \mathbb{R}$.

If α and β are well-ordered lower and upper functions of equation (1.1), then the conditions for the function f are know in the existing literature to guarantee the existence of a solution to problem (1.1), (1.4), as well as problem (1.1), (1.5). More precisely, the following two theorems hold.

Definition 1.2. A continuous function $\omega : \mathbb{R}_+ \to]0,+\infty[$ such that

$$\int\limits_0^{+\infty} \frac{1}{\omega(\eta)}\,\mathrm{d}\eta = +\infty \tag{1.6}$$

is called a Nagumo function.

Theorem 1.1 ([2, Theorem 3.1$_1$ and Remark on p. 2381])**.** *Let α and β be well-ordered lower and upper functions of equation (1.1) and let f satisfy the conditions*

$$f(t,x,y)\operatorname{sgn} y \leq \omega(|y|)\big(h(t)+|y|\big) \ \textit{ for a.e. } \ t \in [a_0,b] \ \textit{ and all } \ x,y \in \mathbb{R}, \ \ \alpha(t) \leq x \leq \beta(t), \ \ |y| \geq R,$$

$$f(t,x,y)\operatorname{sgn} y \geq -\omega(|y|)\big(h(t)+|y|\big) \ \textit{ for a.e. } \ t \in [a,b_0] \ \textit{ and all } \ x,y \in \mathbb{R}, \ \ \alpha(t) \leq x \leq \beta(t), \ \ |y| \geq R,$$

where $a < a_0 < b_0 < b$, $R \geq 0$, $h \in L^1([a,b];\mathbb{R}_+)$, and ω is a Nagumo function. Then, for any $c_1 \in [\alpha(a),\beta(a)]$ and $c_2 \in [\alpha(b),\beta(b)]$, problem (1.1), (1.4) has a solution u satisfying the condition

$$\alpha(t) \leq u(t) \leq \beta(t) \ \textit{ for } \ t \in [a,b]. \tag{1.7}$$

Theorem 1.2 ([2, Theorem 3.1$_2$])**.** *Let α and β be well-ordered lower and upper functions of equation (1.1) such that*

$$\alpha'(a+) \geq \beta'(a+). \tag{1.8}$$

Let, moreover, f satisfy the condition

$$f(t,x,y)\operatorname{sgn} y \leq \omega(|y|)\big(h(t)+|y|\big) \ \textit{ for a.e. } \ t \in [a,b] \ \textit{ and all } \ x,y \in \mathbb{R}, \ \ \alpha(t) \leq x \leq \beta(t), \ \ |y| \geq R, \tag{1.9}$$

where $R \geq 0$, $h \in L^1([a,b];\mathbb{R}_+)$, and ω is a Nagumo function. Then, for any $c_1 \in [\beta'(a+),\alpha'(a+)]$ and $c_2 \in [\alpha(b),\beta(b)]$, problem (1.1), (1.5) has a solution u satisfying (1.7).

2 Notation

Throughout the paper, we use the following notation:

- $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$. For any $x \in \mathbb{R}$, we put $[x]_+ = \frac{1}{2}(|x| + x)$.

- $C(I;\mathbb{R})$ denotes the set of continuous real functions defined on the interval $I \subseteq \mathbb{R}$. For $u \in C([a,b];\mathbb{R})$, we put $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.

- $C^1([a,b];\mathbb{R})$ denotes the Banach space of continuously differentiable functions $u : [a,b] \to \mathbb{R}$ with the norm $\|u\|_{C^1} = \|u\|_C + \|u'\|_C$.

- $AC^1([a,b])$ is the set of functions $u : [a,b] \to \mathbb{R}$ which are absolutely continuous together with their first derivatives.

- $AC_\ell([a,b])$ (resp. $AC_u([a,b])$) is the set of absolutely continuous functions $\gamma : [a,b] \to \mathbb{R}$ such that γ' admits the representation $\gamma'(t) = \gamma_0(t) + \sigma(t)$ for a.e. $t \in [a,b]$, where $\gamma_0 : [a,b] \to \mathbb{R}$ is absolutely continuous and the function σ is non-decreasing (resp. non-increasing) on $[a,b]$ and its derivative is equal to zero almost everywhere on $[a,b]$.

- $L^\nu([a,b];\mathbb{R})$, where $1 \le \nu < +\infty$, is the Banach space of measurable functions $h : [a,b] \to \mathbb{R}$ such that the function $|h|^\nu$ is Lebesgue integrable. For $h \in L^\nu([a,b];\mathbb{R})$, we put $\|h\|_{L^\nu} = \big(\int_a^b |h(s)|^\nu \, \mathrm{d}s\big)^{1/\nu}$.

- $L^\infty([a,b];\mathbb{R})$ is the Banach space of essentially bounded measurable functions $h : [a,b] \to \mathbb{R}$ equipped with the norm $\|h\|_{L^\infty} = \operatorname{esssup}\{|h(t)| : t \in [a,b]\}$.

- Numbers $\mu, \nu \in [1, +\infty]$ are said to be conjugate numbers if $\frac{1}{\mu} + \frac{1}{\nu} = 1$, with $\mu = +\infty$ if $\nu = 1$.

- $Car([a,b] \times A; B)$, where $A \subseteq \mathbb{R}^2$ and $B \subseteq \mathbb{R}$, is the set of functions $f : [a,b] \times A \to B$ such that:

  (1) the function $f(\cdot\,, z) : [a,b] \to B$ is measurable for every $z \in A$,

  (2) the function $f(t, \cdot\,) : A \to B$ is continuous for almost every $t \in [a,b]$,

  (3) for any $r > 0$, there exists $q_r \in L^1([a,b];\mathbb{R}_+)$ such that $|f(t, z_1, z_2)| \le q_r(t)$ for a.e. $t \in [a,b]$ and all $(z_1, z_2) \in A$, $|z_1| + |z_2| \le r$.

- $Car_{sl}([a,b] \times \mathbb{R}_+; \mathbb{R}_+)$ is the set of functions $q \in Car([a,b] \times \mathbb{R}_+; \mathbb{R}_+)$ such that for almost every $t \in [a,b]$, the function $q(t, \cdot\,)$ is non-decreasing on $\mathbb{R}_+$ and

$$\lim_{\varrho \to +\infty} \frac{1}{\varrho} \int_a^b q(s, \varrho) \, \mathrm{d}s = 0.$$

3 Main results

In this section, we provide the main results of the paper; we show their possible use later in Section 4.

Theorem 3.1. *Let α and β be well-ordered lower and upper functions of equation (1.1) such that (1.8) hold and*

$$\alpha(a) = \alpha(b), \quad \beta(a) = \beta(b). \tag{3.1}$$

Let, moreover, f satisfy condition (1.9), where $R \ge 0$, $h \in L^1([a,b];\mathbb{R}_+)$, and ω is a Nagumo function. Then, for any $c \in [\beta'(a), \alpha'(a)]$, problem (1.1), (1.2) has a solution u satisfying (1.7).

Theorem 3.2. *Let α and β be well-ordered lower and upper functions of equation (1.1) such that (1.8) and (3.1) hold and let f satisfy the condition*

$$f(t,x,y)\operatorname{sgn} y \le q(t,|y|)+\sum_{k=1}^{n} h_k(t)|y|^{1+\frac{1}{\mu_k}} \ \text{ for a.e. } t\in[a,b] \text{ and all } x,y\in\mathbb{R}, \alpha(t)\le x\le\beta(t), \ \ |y|\ge R,$$

where $R \ge 0$, $q \in Car_{sl}([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$, and for $k = 1,\dots,n$, $h_k \in L^{\nu_k}([a,b];\mathbb{R}_+)$ and $\mu_k, \nu_k \in [1,+\infty]$ are conjugate numbers (we put $\frac{1}{\mu_k} = 0$ if $\mu_k = +\infty$). Then, for any $c \in [\beta'(a+), \alpha'(a+)]$, problem (1.1), (1.2) has a solution u satisfying (1.7).

Definition 3.1. Let $\alpha : [a,b] \to \mathbb{R}$, $\beta : [a,b] \to \mathbb{R}$ be continuous functions such that (1.3) holds. We say that u^* (resp. u_*) is a maximal solution (resp. a minimal solution) to problem (1.1), (1.2) (or (1.1), (1.4), or (1.1), (1.5)) in the segment $[\alpha,\beta]$ if it is a solution to this problem satisfying $\alpha(t) \le u^*(t) \le \beta(t)$ for $t \in [a,b]$ (resp. $\alpha(t) \le u_*(t) \le \beta(t)$ for $t \in [a,b]$) and for any solution u to problem (1.1), (1.2) (or (1.1), (1.4), or (1.1), (1.5)) with the property (1.7), the inequality $u(t) \le u^*(t)$ for $t \in [a,b]$ (resp. $u(t) \ge u_*(t)$ for $t \in [a,b]$) holds.

Theorem 3.3. *Let the hypotheses of Theorem 3.1 (or Theorem 3.2) be satisfied. Then, for any $c \in [\beta'(a+), \alpha'(a+)]$, problem (1.1), (1.2) has maximal and minimal solutions in the segment $[\alpha,\beta]$.*

Analogously to Theorem 3.3, one can prove the following three statements.

Theorem 3.4. *Let the hypotheses of Theorem 1.1 be satisfied. Then, for any $c_1 \in [\alpha(a), \beta(a)]$ and $c_2 \in [\alpha(b), \beta(b)]$, problem (1.1), (1.4) has maximal and minimal solutions in the segment $[\alpha,\beta]$.*

Theorem 3.5. *Let the hypotheses of Theorem 1.2 be satisfied. Then, for any $c_1 \in [\beta'(a+), \alpha'(a+)]$ and $c_2 \in [\alpha(b), \beta(b)]$, problem (1.1), (1.5) has maximal and minimal solutions in the segment $[\alpha,\beta]$.*

3.1 Auxiliary statements

Lemma 3.1. *Let $u \in C^1([a,b];\mathbb{R})$, $\alpha : [a,b] \to \mathbb{R}$ be absolutely continuous, and $\alpha'(t) = \alpha_0(t) + \sigma(t)$ for a.e. $t \in [a,b]$, where $\alpha_0 \in C([a,b];\mathbb{R})$ and $\sigma : [a,b] \to \mathbb{R}$ is non-decreasing on $[a,b]$. If*

$$u(t) \ge \alpha(t) \ \text{ for } t \in [a,b], \quad u(a) = \alpha(a), \tag{3.2}$$

then $u'(a) \ge \alpha'(a+)$ holds.

Proof. Suppose on the contrary that $u'(a) < \alpha'(a+)$, i.e.,

$$u'(a) - \alpha_0(a) < \sigma(a+). \tag{3.3}$$

Since $u' - \alpha_0 \in C([a,b];\mathbb{R})$, in view of (3.3), there exists $a_0 \in \,]a,b]$ such that

$$u'(t) - \alpha_0(t) < \sigma(a+) \ \text{ for } \ t \in [a,a_0]. \tag{3.4}$$

Moreover, σ is non-decreasing on $[a,b]$ and therefore,

$$\sigma(a+) \le \sigma(t) \ \text{ for } \ t \in \,]a,a_0]. \tag{3.5}$$

Now, it follows from (3.4) and (3.5) that $u'(t) \le \alpha'(t)$ for a.e. $t \in [a,a_0]$ and integrating both parts of this inequality over the interval $[a,a_0]$, we get $u(a_0) - u(a) < \alpha(a_0) - \alpha(a)$, which contradicts (3.2). ☐

The following lemma can be proved in a similar manner.

Lemma 3.2. *Let $u \in C^1([a,b];\mathbb{R})$, $\beta : [a,b] \to \mathbb{R}$ be absolutely continuous, and $\beta'(t) = \beta_0(t) + \sigma(t)$ for a.e. $t \in [a,b]$, where $\beta_0 \in C([a,b];\mathbb{R})$ and $\sigma : [a,b] \to \mathbb{R}$ is non-increasing on $[a,b]$. If $u(t) \le \beta(t)$ for $t \in [a,b]$, $u(a) = \beta(a)$, then $u'(a) \le \beta'(a+)$ holds.*

Lemma 3.3. *Let α and β be well-ordered lower and upper functions of equation* (1.1) *such that* (1.8) *and* (3.1) *hold and let f satisfy the condition*

$$|f(t,x,y)| \le \ell(t,|x|+|y|) \ \text{ for a.e. } t \in [a,b] \ \text{ and all } x,y \in \mathbb{R}, \ \alpha(t) \le x \le \beta(t), \tag{3.6}$$

where $\ell \in Car_{sl}([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$. Then, for any $c \in [\beta'(a+),\alpha'(a+)]$, problem (1.1), (1.2) *has a solution u satisfying* (1.7).

Proof. Let $c \in [\beta'(a+),\alpha'(a+)]$ be arbitrary. It is well known that, under the assumption (3.6), for any $\lambda \in [\alpha(a),\beta(a)]$, the Dirichlet problem

$$u'' = f(t,u,u'); \quad u(a)=\lambda, \ \ u(b)=\lambda \tag{3.7_λ}$$

has a solution u satisfying (1.7). Put

$$\mathcal{U} \overset{\text{def}}{=} \big\{u : \text{ there exists } \lambda \in [\alpha(a),\beta(a)] \text{ such that } u \text{ is a solution to } (3.7_\lambda) \text{ satisfying } (1.7)\big\}.$$

Clearly, $\mathcal{U} \neq \varnothing$. We show that

$$\text{the set } \mathcal{U} \text{ is compact in the space } C^1([a,b];\mathbb{R}). \tag{3.8}$$

Let $u \in \mathcal{U}$ be arbitrary. It is clear that $\|u\|_C \le \max\{|\alpha(t)|+|\beta(t)| : \ t \in [a,b]\} \le \|\alpha\|_C + \|\beta\|_C$ and there exists $t_0 \in [a,b]$ such that $u'(t_0)=0$ and thus, in view of (3.6), the solution u to equation (1.1) satisfies

$$\begin{aligned}|u'(t)| &\le \left|\int_{t_0}^{t} |f(s,u(s),u'(s))|\,\mathrm{d}s\right| \le \int_a^b \ell(s,|u(s)|+|u'(s)|)\,\mathrm{d}s \\ &\le \int_a^b \ell(s,\|u\|_C+\|u'\|_C)\,\mathrm{d}s \le \int_a^b \ell\big(s,\|\alpha\|_C+\|\beta\|_C+\|u'\|_C\big)\,\mathrm{d}s\end{aligned}$$

for $t \in [a,b]$. Consequently, we get

$$\|u'\|_C \le \int_a^b \ell\big(s,\|\alpha\|_C+\|\beta\|_C+\|u'\|_C\big)\,\mathrm{d}s. \tag{3.9}$$

Thus, we have proved that for every $u \in \mathcal{U}$, the relations

$$\|u\|_C \le \|\alpha\|_C + \|\beta\|_C, \quad \|u'\|_C \le \int_a^b \ell\big(s,\|\alpha\|_C+\|\beta\|_C+\|u'\|_C\big)\,\mathrm{d}s \tag{3.10}$$

are satisfied. Since $\ell \in Car_{sl}([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$, there exists $r_0>0$ such that

$$\int_a^b \ell(s,\Delta+z)\,\mathrm{d}s < z \ \text{ for } \ z \ge r_0,$$

where $\Delta \overset{\text{def}}{=} \|\alpha\|_C + \|\beta\|_C$. Hence, in view of the second relation in (3.10), we get

$$\|u'\|_C \le r_0 \ \text{ for every } \ u \in \mathcal{U}. \tag{3.11}$$

Having proven estimates (3.10) and (3.11), using the standard arguments, we easily show that assertion (3.8) holds.

We now put

$$\mathcal{A} \overset{\text{def}}{=} \big\{u(a) : \ u \in \mathcal{U}, \ u'(a) \ge c\big\}. \tag{3.12}$$

According to the above proved, there exists a solution $u_{\alpha(a)}$ to problem ($3.7_{\alpha(a)}$) satisfying $\alpha(t) \le u_{\alpha(a)}(t) \le \beta(t)$ for $t \in [a,b]$. Since $u_{\alpha(a)}(a) = \alpha(a)$ and $u_{\alpha(a)}(t) \ge \alpha(t)$ for $t \in [a,b]$, Lemma 3.1 yields that $u'_{\alpha(a)}(a) \ge \alpha'(a+) \ge c$ and thus, $\alpha(a) \in \mathcal{A}$. Therefore, $\mathcal{A} \neq \varnothing$. It follows from (3.8) that the set $\mathcal{A}$ is compact in $[\alpha(a), \beta(a)]$, which allows one to put

$$\lambda^* \overset{\text{def}}{=} \max \mathcal{A}. \tag{3.13}$$

Let $u_{\lambda^*} \in \mathcal{U}$ be such that

$$u_{\lambda^*}(a) = \lambda^*, \quad u'_{\lambda^*}(a) \ge c. \tag{3.14}$$

First, we assume that $\lambda^* = \beta(a)$. Then, $u_{\lambda^*}(a) = \lambda^* = \beta(a)$ and $u_{\lambda^*}(t) \le \beta(t)$ for $t \in [a,b]$, which guarantees that $u'_{\lambda^*}(a) \le \beta'(a+) \le c$ (see Lemma 3.2). On the other hand, (3.14) holds and therefore, $u'_{\lambda^*}(a) = c$. This means that u_{λ^*} is the desired solution to problem (1.1), (1.2) and thus, the lemma is proved.

Now, assume that $\lambda^* < \beta(a)$. In the same manner as above (we only use u_{λ^*} instead of α), one can prove that for any $\lambda \in [\lambda^*, \beta(a)]$, there exists a solution v to problem (3.7_λ) satisfying

$$u_{\lambda^*}(t) \le v(t) \le \beta(t) \ \text{ for } \ t \in [a,b]. \tag{3.15}$$

Put

$$\mathcal{B} \overset{\text{def}}{=} \big\{ v(a) : \ v \in \mathcal{U}, \ (3.15) \text{ holds}, \ v'(a) \le c \big\}. \tag{3.16}$$

It is clear that there exists a solution $v_{\beta(a)}$ to problem ($3.7_{\beta(a)}$) satisfying $u_{\lambda^*}(t) \le v_{\beta(a)}(t) \le \beta(t)$ for $t \in [a,b]$. Since $v_{\beta(a)}(a) = \beta(a)$ and $v_{\beta(a)}(t) \le \beta(t)$ for $t \in [a,b]$, it follows from Lemma 3.2 that $v'_{\beta(a)}(a) \le \beta'(a+) \le c$ and thus, $\beta(a) \in \mathcal{B}$. Therefore, $\mathcal{B} \neq \varnothing$. It follows from (3.8) that the set $\mathcal{B}$ is compact in $[\lambda^*, \beta(a)]$, which allows one to put

$$\lambda_* = \min \mathcal{B}. \tag{3.17}$$

Let $v_{\lambda^*} \in \mathcal{U}$ be such that

$$u_{\lambda^*}(t) \le v_{\lambda_*}(t) \le \beta(t) \ \text{ for } \ t \in [a,b], \quad v_{\lambda_*}(a) = \lambda_*, \ \ v'_{\lambda_*}(a) \le c. \tag{3.18}$$

First, we mention that, by (3.12), (3.13), (3.16), and (3.17), the relation $\lambda^* \le \lambda_*$ holds. We will show that

$$\lambda^* = \lambda_*. \tag{3.19}$$

Suppose on the contrary that $\lambda^* < \lambda_*$. Then there exist $\lambda_0 \in]\lambda^*, \lambda_*[$ and a solution v_{λ_0} to problem (3.7_{λ_0}) satisfying $u_{\lambda^*}(t) \le v_{\lambda_0}(t) \le \beta(t)$ for $t \in [a,b]$. Since $\lambda_0 < \lambda_*$, in view of (3.17), we get $\lambda_0 \notin \mathcal{B}$ which yields

$$v'_{\lambda_0}(a) > c. \tag{3.20}$$

On the other hand, on account of (3.13), the inequality $\lambda_0 > \lambda^*$ implies $v'_{\lambda_0}(a) < c$ that contradicts (3.20). The obtained contradiction proves equality (3.19). Therefore, it follows from (3.14) and (3.18) that $u_{\lambda^*}(a) = \lambda^* = \lambda_* = v_{\lambda_*}(a)$ and $u'_{\lambda^*}(a) = v'_{\lambda_*}(a) = c$. This means that u_{λ^*} (as well as v_{λ_*}) is the desired solution to problem (1.1), (1.2), and thus the lemma is proved. □

Lemma 3.4. *Let $c \in \mathbb{R}$, $r > 0$, $R \ge 0$, $h \in L^1([a,b]; \mathbb{R}_+)$, and ω be a Nagumo function. Then there exists $\varrho > 0$ such that for any $\widetilde{f} \in Car([a,b] \times \mathbb{R}^2; \mathbb{R})$ satisfying*

$$\widetilde{f}(t,x,y) \operatorname{sgn} y \le \omega(|y|)\big(h(t) + |y|\big) \ \text{ for a.e. } \ t \in [a,b] \ \text{ and all } \ x, y \in \mathbb{R}, \ \ |x| \le r, \ \ |y| \ge R \tag{3.21}$$

and for every solution u to the problem

$$u'' = \widetilde{f}(t,u,u'); \quad u'(a) = c, \ \ u(a) = u(b) \tag{3.22}$$

satisfying $\|u\|_C \le r$, the inequality

$$\|u'\|_C \le \varrho \tag{3.23}$$

holds.

Proof. It follows from (1.6) that there exists $\varrho \geq \max\{|c|, R\}$ such that

$$\int\limits_{\max\{|c|,R\}}^{z} \frac{1}{\omega(\eta)}\,\mathrm{d}\eta > \|h\|_{L^1} + 2r \quad \text{for} \quad z \geq \varrho. \tag{3.24}$$

Let $\widetilde{f} \in \mathit{Car}([a,b] \times \mathbb{R}^2; \mathbb{R})$ satisfying (3.21) be arbitrary and let u be a solution to problem (3.22) such that $\|u\|_C \leq r$.

Choose $t_0 \in \,]a,b]$ such that $|u'(t_0)| = \|u'\|_C$. If $|u'(t_0)| \leq \max\{|c|, R\}$, then (3.23) is clearly fulfilled. Suppose that $|u'(t_0)| > \max\{|c|, R\}$. Since $|u'(a)| = |c| \leq \max\{|c|, R\}$, there exists $a_0 \in [a, t_0[$ such that

$$|u'(t)| > \max\big\{|c|, R\big\} \quad \text{for} \quad t \in [a_0, t_0], \qquad |u'(a_0)| = \max\big\{|c|, R\big\}. \tag{3.25}$$

Moreover, $\|u\|_C \leq r$, and thus, by virtue of (3.21), we get

$$|u'(t)|' = u''(t) \operatorname{sgn} u'(t) \leq \omega(|u'(t)|)\big(h(t) + |u'(t)|\big) \quad \text{for a.e.} \quad t \in [a_0, t_0].$$

Hence,

$$\frac{|u'(t)|'}{\omega(|u'(t)|)} \leq h(t) + |u'(t)| \quad \text{for a.e.} \quad t \in [a_0, t_0],$$

and integrating this inequality over the interval $[a_0, t_0]$, we get

$$\int\limits_{|u'(a_0)|}^{|u'(t_0)|} \frac{1}{\omega(\eta)}\,\mathrm{d}\eta \leq \int\limits_{a_0}^{t_0} h(s)\,\mathrm{d}s + |u(t_0)| + |u(a_0)|. \tag{3.26}$$

In view of (3.25), the assumption $\|u\|_C \leq r$, and the choice of t_0, inequality (3.26) yields

$$\int\limits_{\max\{|c|,R\}}^{\|u'\|_C} \frac{1}{\omega(\eta)}\,\mathrm{d}\eta \leq \|h\|_{L^1} + 2r$$

which, using (3.24), leads to (3.23). □

Lemma 3.5. *Let $c \in \mathbb{R}$, $r > 0$, $R \geq 0$, $q \in \mathit{Car}_{sl}([a,b] \times \mathbb{R}_+; \mathbb{R}_+)$, $\nu_k, \mu_k \in [1, +\infty]$ be conjugate numbers, and $h_k \in L^{\nu_k}([a,b]; \mathbb{R}_+)$ for $k = 1, \dots, n$. Then there exists $\varrho > 0$ such that for any $\widetilde{f} \in \mathit{Car}([a,b] \times \mathbb{R}^2; \mathbb{R})$ satisfying*

$$\widetilde{f}(t,x,y) \operatorname{sgn} y \leq q(t,|y|) + \sum_{k=1}^{n} h_k(t) |y|^{1+\frac{1}{\mu_k}} \quad \text{for a.e.} \quad t \in [a,b] \quad \text{and all} \quad x, y \in \mathbb{R}, \quad |x| \leq r, \quad |y| \geq R \tag{3.27}$$

(we put $\frac{1}{\mu_k} = 0$ if $\mu_k = +\infty$) and for every solution u to problem (3.22) satisfying $\|u\|_C \leq r$, inequality (3.23) holds.

Proof. It follows from the sub-linearity of q at infinity that there is $\varrho \geq \max\{|c|, R\}$ such that

$$\bigg(\max\big\{|c|, R\big\} + \int\limits_a^b q(s,z)\,\mathrm{d}s\bigg) \exp\bigg(\sum_{k=1}^{n} \|h_k\|_{L^{\nu_k}} (2r)^{\frac{1}{\mu_k}}\bigg) < z \quad \text{for} \quad z \geq \varrho. \tag{3.28}$$

Let $\widetilde{f} \in \mathit{Car}([a,b] \times \mathbb{R}^2; \mathbb{R})$ satisfying (3.27) be arbitrary and let u be a solution to problem (3.22) such that $\|u\|_C \leq r$.

Choose $t_0 \in \,]a,b]$ such that $|u'(t_0)| = \|u'\|_C$. If $|u'(t_0)| \leq \max\{|c|, R\}$, then (3.23) is clearly fulfilled. Suppose that $|u'(t_0)| > \max\{|c|, R\}$. Since $|u'(a)| = |c| \leq \max\{|c|, R\}$, there exists $a_0 \in [a, t_0[$ such that (3.25) holds. Moreover, $\|u\|_C \leq r$, and thus, by virtue of (3.27), we get

$$|u'(t)|' = u''(t)\operatorname{sgn} u'(t) \le q(t, |u'(t)|) + \sum_{k=1}^{n} h_k(t)|u'(t)|^{1+\frac{1}{\mu_k}}$$
$$\le q(t, \|u'\|_C) + |u'(t)| \sum_{k=1}^{n} h_k(t)|u'(t)|^{\frac{1}{\mu_k}} \text{ for a.e. } t \in [a_0, t_0]$$

(here, we put $|u'(t)|^{\frac{1}{\mu_k}} = 1$ if $\mu_k = +\infty$). Hence, in view of (3.25), we have

$$|u'(t)| \le |u'(a_0)| + \int_{a_0}^{t} q(s, \|u'\|_C)\,\mathrm{d}s + \int_{a_0}^{t} \Big(\sum_{k=1}^{n} h_k(s)|u'(s)|^{\frac{1}{\mu_k}}\Big)|u'(s)|\,\mathrm{d}s$$
$$\le \max\{|c|, R\} + \int_{a}^{b} q(s, \|u'\|_C)\,\mathrm{d}s + \int_{a_0}^{t} \Big(\sum_{k=1}^{n} h_k(s)|u'(s)|^{\frac{1}{\mu_k}}\Big)|u'(s)|\,\mathrm{d}s$$

for $t \in [a_0, t_0]$. The Gronwall–Bellman lemma then yields

$$|u'(t_0)| \le \Big(\max\big\{|c|, R\big\} + \int_{a}^{b} q(s, \|u'\|_C)\,\mathrm{d}s\Big) \exp\Big(\int_{a_0}^{t_0} \Big(\sum_{k=1}^{n} h_k(s)|u'(s)|^{\frac{1}{\mu_k}}\Big)\,\mathrm{d}s\Big). \tag{3.29}$$

Using the Hölder inequality, we obtain

$$\int_{a_0}^{t_0} \Big(\sum_{k=1}^{n} h_k(s)|u'(s)|^{\frac{1}{\mu_k}}\Big)\,\mathrm{d}s = \sum_{k=1}^{n} \int_{a_0}^{t_0} h_k(s)|u'(s)|^{\frac{1}{\mu_k}}\,\mathrm{d}s$$
$$\le \sum_{k=1}^{n} \|h_k\|_{L^{\nu_k}([a_0,t_0];\mathbb{R})} \Big(\int_{a_0}^{t_0} |u'(s)|\,\mathrm{d}s\Big)^{\frac{1}{\mu_k}} \le \sum_{k=1}^{n} \|h_k\|_{L^{\nu_k}} \big(|u(t_0)| + |u(a_0)|\big)^{\frac{1}{\mu_k}} \le \sum_{k=1}^{n} \|h_k\|_{L^{\nu_k}} (2r)^{\frac{1}{\mu_k}}$$

(here, we put $z^{\frac{1}{\mu_k}} = 1$ if $\mu_k = +\infty$). Consequently, it follows from (3.29) that

$$\|u'\|_C = |u'(t_0)| \le \Big(\max\{|c|, R\} + \int_{a}^{b} q(s, \|u'\|_C)\,\mathrm{d}s\Big) \exp\Big(\sum_{k=1}^{n} \|h_k\|_{L^{\nu_k}} (2r)^{\frac{1}{\mu_k}}\Big)$$

which, using (3.28), leads to (3.23). □

3.2 Proofs of the main results

Proof of Theorem 3.1. Let $c \in [\beta'(a), \alpha'(a)]$ be arbitrary and let ϱ be the number appearing in Lemma 3.4 with $r \overset{\text{def}}{=} \|\alpha\|_C + \|\beta\|_C$. Put

$$\Delta \overset{\text{def}}{=} \varrho + \operatorname{esssup}\big\{|\alpha'(t)| + |\beta'(t)| :\ t \in [a, b]\big\}, \tag{3.30}$$

$$\chi_{\alpha\beta}(t, x) \overset{\text{def}}{=} x + [\alpha(t) - x]_+ - [x - \beta(t)]_+ \text{ for } t \in [a, b],\ x \in \mathbb{R},$$

$$\chi_{\Delta}(y) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } |y| \le \Delta, \\ 2 - \dfrac{|y|}{\Delta} & \text{if } \Delta < |y| \le 2\Delta, \\ 0 & \text{if } |y| \ge 2\Delta \end{cases} \quad \text{for } y \in \mathbb{R}, \tag{3.31}$$

and $\widetilde{f}(t, x, y) \overset{\text{def}}{=} \chi_{\Delta}(y) f(t, \chi_{\alpha\beta}(t, x), y)$ for a.e. $t \in [a, b]$ and all $x, y \in \mathbb{R}$. It is clear that

$$-\|\alpha\|_C \le \alpha(t) \le \chi_{\alpha\beta}(t, x) \le \beta(t) \le \|\beta\|_C \text{ for } t \in [a, b],\ x \in \mathbb{R}, \tag{3.32}$$
$$0 \le \chi_{\Delta}(y) \le 1 \text{ for } y \in \mathbb{R}, \tag{3.33}$$

and there exists $q^* \in L^1([a,b];\mathbb{R})$ such that the function $\widetilde{f} \in Car([a,b]\times\mathbb{R}^2;\mathbb{R})$ satisfies $|\widetilde{f}(t,x,y)| \le q^*(t)$ for a.e. $t \in [a,b]$ and all $x,y \in \mathbb{R}$. Moreover, in view of (3.30), α and β are the lower and upper functions of the equation

$$u'' = \widetilde{f}(t,u,u') \tag{3.34}$$

and thus, it follows from Lemma 3.3 with $\ell(t,z) \overset{\text{def}}{=} q^*(t)$ that problem (3.34), (1.2) has a solution u satisfying (1.7). Obviously, $\chi_{\alpha\beta}(t,u(t)) = u(t)$ for $t \in [a,b]$. On the other hand, in view of (3.32) and (3.33), condition (1.9) yields

$$\begin{aligned}\widetilde{f}(t,x,y)\operatorname{sgn} y &= \chi_\Delta(y) f(t,\chi_{\alpha\beta}(t,x),y)\operatorname{sgn} y \\ &\le \chi_\Delta(y)\omega(y)\big(h(t)+|y|\big) \le \omega(y)\big(h(t)+|y|\big) \ \text{ for a.e. } t\in[a,b] \text{ and all } x,y\in\mathbb{R},\ |y|\ge R.\end{aligned}$$

Since $\|u\|_C \le \|\alpha\|_C + \|\beta\|_C$, Lemma 3.4 with $r \overset{\text{def}}{=} \|\alpha\|_C + \|\beta\|_C$ guarantees that $\|u'\|_C \le \varrho \le \Delta$, and thus $\chi_\Delta(u'(t)) = 1$ for $t \in [a,b]$. Consequently, u is also a solution to equation (1.1). □

Proof of Theorem 3.2. The proof is analogous to that of Theorem 3.1; the only difference is that we apply Lemma 3.5 instead of Lemma 3.4. □

Proof of Theorem 3.3. Assume that the hypotheses of Theorem 3.1 (resp. Theorem 3.2) are fulfilled and let $c \in [\beta'(a+),\alpha'(a+)]$ be arbitrary. Put

$$\mathcal{U}_c \overset{\text{def}}{=} \big\{u:\ u \text{ is a solution to problem (1.1), (1.2) satisfying (1.7)}\big\}.$$

Theorem 3.1 (resp. Theorem 3.2) guarantees that $\mathcal{U}_c \neq \varnothing$. Obviously, $\|u\|_C \le \|\alpha\|_C + \|\beta\|_C$ for every $u \in \mathcal{U}_c$. Moreover, it follows from Lemma 3.4 (resp. Lemma 3.5) with $r \overset{\text{def}}{=} \|\alpha\|_C + \|\beta\|_C$ and $\widetilde{f}(t,x,y) \overset{\text{def}}{=} f(t,\chi_{\alpha\beta}(t,x),y)$, where $\chi_{\alpha\beta}$ is given by (3.31), that there exists $\varrho > 0$ such that $\|u'\|_C \le \varrho$ for every $u \in \mathcal{U}_c$. Consequently, using the standard arguments, we easily show that the set $\mathcal{U}_c$ is compact in the space $C^1([a,b];\mathbb{R})$.

Define the functional

$$F(u) \overset{\text{def}}{=} \int_a^b u(s)\,\mathrm{d}s \ \text{ for } \ u \in C^1([a,b];\mathbb{R}).$$

This functional is evidently continuous, and thus it attains maximal and minimal values on the compact set $\mathcal{U}_c$ at some elements $u_M \in \mathcal{U}_c$ and $u_m \in \mathcal{U}_c$, respectively. We show that u_M is a maximal solution to problem (1.1), (1.2) in the segment $[\alpha,\beta]$.

Suppose on the contrary that u_0 is a solution to (1.1), (1.2) such that

$$\alpha(t) \le u_0(t) \le \beta(t) \ \text{ for } \ t \in [a,b], \quad u_0(t_0) > u_M(t_0) \ \text{ for some } \ t_0 \in [a,b]. \tag{3.35}$$

Then there are three possibilities:

If $u_0(t) \ge u_M(t)$ for $t \in [a,b]$, we put $\alpha_0(t) \overset{\text{def}}{=} u_0(t)$ for $t \in [a,b]$.

If $\min\{u_0(t) - u_M(t) : t \in [a,b]\} < 0$ and $u_0(a) > u_M(a)$, we put

$$\begin{aligned} a_0 &\overset{\text{def}}{=} \sup\big\{t\in[a,b]:\ u_0(s) > u_M(s) \text{ for } s\in[a,t]\big\},\\ b_0 &\overset{\text{def}}{=} \inf\big\{t\in[a,b]:\ u_0(s) > u_M(s) \text{ for } s\in[t,b]\big\}.\end{aligned}$$

Then $a < a_0 < b_0 < b$,

$$u_0(t) > u_M(t) \ \text{ for } \ t \in [a,a_0[\,\cup\,]b_0,b], \tag{3.36}$$

and $u_0(a_0) = u_M(a_0)$, $u_M(b_0) = u_0(b_0)$, $u_0'(a_0) \le u_M'(a_0)$, $u_M'(b_0) \le u_0'(b_0)$. In this case, we define the function α_0 by the formula

$$\alpha_0(t) \overset{\text{def}}{=} \begin{cases} u_0(t) & \text{for } t \in [a,a_0[\,\cup\,]b_0,b], \\ u_M(t) & \text{for } t \in [a_0,b_0]. \end{cases}$$

If $\min\{u_0(t) - u_M(t) :\ t \in [a,b]\} < 0$ and $u_0(a) \le u_M(a)$, then $t_0 \in\,]a,b[$, and we put

$$a_0 \overset{\text{def}}{=} \inf\big\{t \in [a,t_0] :\ u_0(s) > u_M(s) \text{ for } s \in [t,t_0]\big\},$$
$$b_0 \overset{\text{def}}{=} \sup\big\{t \in [t_0,b] :\ u_0(s) > u_M(s) \text{ for } s \in [t_0,t]\big\}.$$

Then $a \le a_0 < b_0 \le b$,

$$\begin{gathered} u_0(t) > u_M(t) \ \text{ for } \ t \in\,]a_0,b_0[\,, \\ u_0(a_0) = u_M(a_0), \quad u_M(b_0) = u_0(b_0), \end{gathered} \tag{3.37}$$

and $u_M'(a_0) \le u_0'(a_0)$, $u_0'(b_0) \le u_M'(b_0)$ if $b_0 < b$. In this case, we define the function α_0 by the formula

$$\alpha_0(t) \overset{\text{def}}{=} \begin{cases} u_M(t) & \text{for } \ t \in [a,a_0], \\ u_0(t) & \text{for } \ t \in\,]a_0,b_0[\,, \\ u_M(t) & \text{for } \ t \in [b_0,b]. \end{cases}$$

One can show that, in all three cases, we have $\alpha_0 \in AC_\ell([a,b];\mathbb{R})$,

$$\begin{gathered} \alpha_0'(a) = c, \quad \alpha_0(a) = \alpha_0(b), \\ \alpha_0''(t) = f(t,\alpha_0(t),\alpha_0'(t)) \ \text{ for a.e. } \ t \in [a,b], \\ \alpha_0(t) \le \beta(t) \ \text{ for } \ t \in [a,b], \end{gathered}$$

and, moreover, (3.35), (3.36), and (3.37) lead to

$$\max\big\{\alpha_0(t) - u_M(t) :\ t \in [a,b]\big\} > 0. \tag{3.38}$$

Therefore, it follows from Theorem 3.1 (resp. Theorem 3.2) with $\alpha(t) \overset{\text{def}}{=} \alpha_0(t)$ that problem (1.1), (1.2) has a solution $\widetilde{u}$ satisfying

$$u_M(t) \le \alpha_0(t) \le \widetilde{u}(t) \le \beta(t) \ \text{ for } \ t \in [a,b], \tag{3.39}$$

and thus $\widetilde{u} \in \mathcal{U}_c$. However, in view of (3.38) and (3.39), we get $F(\widetilde{u}) > F(u_M)$, which contradicts the fact that F attains its maximal value on $\mathcal{U}_c$ at u_M.

One can show in a similar manner that u_m is a minimal solution to problem (1.1), (1.2) in the segment $[\alpha,\beta]$. □

4 Applications of main results

In this section, we provide the consequences of the main results for a pendulum inside a closed box, which moves in a plane. More precisely, consider a free damped pendulum consisting of a small solid of the weight m attached to the massless rod of the length ℓ (see Fig. 1).

The pendulum swings inside a closed box filled with a fluid; the given functions φ and ψ determine box's position in time. This is a system with one degree of freedom, described by the generalized coordinate θ, whose Lagrangian is of the form

$$L(\theta,\dot\theta,t) \overset{\text{def}}{=} \frac{m}{2}\Big[(\dot\varphi(t) + \ell\dot\theta\cos\theta)^2 + (\dot\psi(t) - \ell\dot\theta\sin\theta)^2\Big] - mg(\psi(t) + \ell\cos\theta),$$

because for the x,y coordinates of the solid, we have $x = \varphi(t) + \ell\sin\theta$ and $y = \psi(t) + \ell\cos\theta$. The pendulum moves throughout a fluid; neglecting compressive forces on the solid's surface and considering the simplest model of fluid friction, the viscous friction force is given by the formula

$$\vec F_{\text{fric}} = -\mu\vec v_{\text{rel}} = -\mu(\dot x - \dot\varphi(t), \dot y - \dot\psi(t)) = -\mu\ell\dot\theta(-\cos\theta, \sin\theta),$$

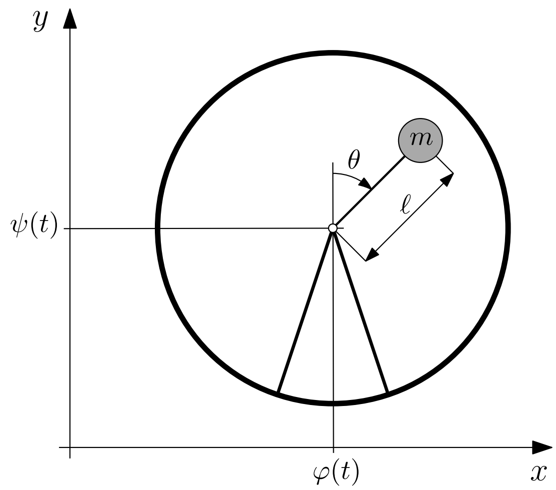

Figure 1

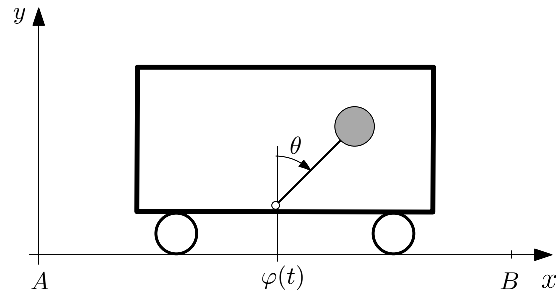

Figure 2

where μ is a viscous friction coefficient and $\vec{v}_{\rm rel}$ is the relative velocity between the fluid and solid. Therefore, the corresponding generalized force is

$$Q = \vec{F}_{\rm fric} \cdot \frac{\partial}{\partial\theta}(x,y) = \vec{F}_{\rm fric} \cdot \frac{\partial}{\partial\theta}(\varphi(t) + \ell\sin\theta,$$
$$y = \psi(t) + \ell\cos\theta) = -\mu\ell\dot{\theta}(\cos\theta, -\sin\theta)\cdot(\ell\cos\theta, -\ell\sin\theta) = -\mu\ell^2\dot{\theta}.$$

Consequently, the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t} L_{\dot{\theta}} - L_\theta = Q$$

leads to the equation of motion

$$\ddot{\theta} + \frac{\mu}{m}\,\dot{\theta} + \frac{\ddot{\varphi}(t)}{\ell}\,\cos\theta - \Big(\frac{\mathrm{g}}{\ell} + \frac{\ddot{\psi}(t)}{\ell}\Big)\sin\theta = 0. \tag{4.1}$$

It is worth mentioning here that if we consider another model of fluid friction, namely, $\vec{F}_{\rm fric} = -\mu|\vec{v}_{\rm rel}|\vec{v}_{\rm rel}$, then (4.1) involves the term $\frac{\mu\ell}{m}|\dot{\theta}|\dot{\theta}$ instead of $\frac{\mu}{m}\,\dot{\theta}$.

Courant and Robbins in their book “What is Mathematics” (see [1]) formulated a problem posed by H. Whitney. The problem is stated as follows. “Suppose a train travels from station A to station B along a straight section of track. The journey need not be of uniform speed or acceleration. The train may act in any manner, speeding up, slowing down, coming to a halt, or even backing up for a while, before reaching B. But the exact motion of the train is supposed to be known in advance; that is, the function $s = \varphi(t)$ is given, where s is the distance of the train from station A, and t is the time, measured from the instant of departure. On the floor of one of the cars, a rod is pivoted so that it may move without friction either forward or backward until it touches the floor. If it does touch the floor, we assume that it remains on the floor henceforth; this will be the case if the rod does not bounce. Is it possible to place the rod in such a position that, if it is released at the instant when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey from A to B?” It is clear that, assuming a point mass m is attached to the massless rod instead of the rod itself (see Fig. 2), a motion of the rod is determined by a solution to the equation

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell}\,\cos\theta - \frac{\mathrm{g}}{\ell}\,\sin\theta = 0$$

on the interval $[0,T]$ satisfying the condition

$$-\frac{\pi}{2} < \theta(t) < \frac{\pi}{2} \;\text{ for }\; t \in [0,T]. \tag{4.2}$$

Inspired by the above problems, we now apply the main results and provide the effective conditions for the existence as well as uniqueness of a solution u to the equation

$$u'' = p(t)\sin u + g(t)\cos u + h(t)|u'|^{\lambda}\operatorname{sgn} u'(t), \tag{4.3}$$

satisfying the boundary conditions of types (1.2), (1.4), and (1.5), and the inequalities $-\frac{\pi}{2} < u(t) < \frac{\pi}{2}$ for $t \in]a,b[$. In what follows, we assume that $p, g, h \in L^1([a,b];\mathbb{R})$ and $\lambda \geq 1$.

Let us start with a lemma that guarantees, in fact, that if $p(t) \geq 0$, then no non-constant solution to equation (4.3) can "touch $-\frac{\pi}{2}$ and $\frac{\pi}{2}$".

Lemma 4.1. *Let $\lambda \geq 1$, $a \leq t_1 < t_2 \leq b$,*

$$p(t) \geq 0 \;\; \text{for a.e.} \;\; t \in [t_1, t_2], \tag{4.4}$$

and u be a solution to equation (4.3) satisfying

$$-\frac{\pi}{2} \leq u(t) \leq \frac{\pi}{2} \;\; \text{for} \;\; t \in [t_1, t_2], \tag{4.5}$$

$$u'(t_1) = 0, \;\; u(t_2) \in \Big] -\frac{\pi}{2}, \frac{\pi}{2}\Big[\;\; \Big(\text{resp.} \;\; u(t_1) \in \Big] -\frac{\pi}{2}, \frac{\pi}{2}\Big[, \;\; u'(t_2) = 0\Big),$$

then $-\frac{\pi}{2} < u(t) < \frac{\pi}{2}$ for $t \in [t_1, t_2]$.

Proof. Assume that $u'(t_1) = 0$ and $u(t_2) \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ (if $u(t_1) \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $u'(t_2) = 0$, the proof is analogous).

We first show that

$$u(t) < \frac{\pi}{2} \;\; \text{for} \;\; t \in [t_1, t_2]. \tag{4.6}$$

Indeed, suppose on the contrary that (4.6) does not hold. Then there exists $t_0 \in [t_1, t_2[$ such that

$$u(t_0) = \frac{\pi}{2}, \tag{4.7}$$

which, together with (4.5) and the assumption $u'(t_1) = 0$, yields

$$u'(t_0) = 0. \tag{4.8}$$

Moreover, in view of (4.5) and (4.7), there exists $t^* \in]t_0, t_2]$ such that

$$0 \leq u(t) \leq \frac{\pi}{2} \;\; \text{for} \;\; t \in [t_0, t^*], \quad u(t^*) < \frac{\pi}{2}. \tag{4.9}$$

Using (4.4) and (4.9), from (4.3) we get

$$-u''(t) = -p(t)\sin u(t) - g(t)\cos u(t) - h(t)|u'(t)|^{\lambda-1}u'(t) \leq |g(t)|\cos u(t) - h(t)|u'(t)|^{\lambda-1}u'(t) \quad \text{for a.e.} \;\; t \in [t_0, t^*]$$

(we put $|u'(t)|^{\lambda-1} = 1$ if $\lambda = 1$), and thus, for a.e. $t \in [t_0, t^*]$, the inequality

$$\Big(u'(t)\,\mathrm{e}^{-\int_{t_0}^{t} h(\eta)|u'(\eta)|^{\lambda-1}\,\mathrm{d}\eta}\Big)' \leq |g(t)|\,\mathrm{e}^{-\int_{t_0}^{t} h(\eta)|u'(\eta)|^{\lambda-1}\,\mathrm{d}\eta}\cos u(t)$$

holds.

Integrating this inequality over the interval $[t_0, t]$ and taking into account (4.8) and the inequality $\cos x \leq \frac{\pi}{2} - x$ for $x \in [0, \frac{\pi}{2}]$, we arrive at

$$\begin{aligned}\Big(\frac{\pi}{2} - u(t)\Big)' = -u'(t) &\leq \int_{t_0}^{t} |g(s)|\,\mathrm{e}^{-\int_{s}^{t} h(\eta)|u'(\eta)|^{\lambda-1}\,\mathrm{d}\eta}\cos u(s)\,\mathrm{d}s \\ &\leq \exp\big(\|h\|_{L^1}\|u'\|_C^{\lambda-1}\big)\int_{t_0}^{t} |g(s)|\Big(\frac{\pi}{2} - u(s)\Big)\,\mathrm{d}s \;\; \text{for} \;\; t \in [t_0, t^*].\end{aligned}$$

Integrating this inequality over the interval $[t_0, t]$ once more and using (4.5) and (4.7), we obtain

$$0 \le \frac{\pi}{2} - u(t) \le \exp\left(\|h\|_{L^1}\|u'\|_C^{\lambda-1}\right) \int_{t_0}^{t} \left(\int_{t_0}^{s} |g(\xi)| \left(\frac{\pi}{2} - u(\xi) \right) \mathrm{d}\xi \right) \mathrm{d}s$$

$$\le (t^* - t_0) \exp\left(\|h\|_{L^1}\|u'\|_C^{\lambda-1}\right) \int_{t_0}^{t} |g(s)| \left(\frac{\pi}{2} - u(s) \right) \mathrm{d}s \quad \text{for } t \in [t_0, t^*].$$

Therefore, the Gronwall–Bellman lemma yields $\frac{\pi}{2} - u(t^*) \le 0$, which contradicts (4.9). The obtained contradiction proves that inequality (4.6) holds.

On the other hand, the inequality $u(t) > -\frac{\pi}{2}$ for $t \in [t_1, t_2]$ follows from the above-proved inequality (4.6) and the fact that $-u$ is a solution to equation

$$u'' = p(t) \sin u - g(t) \cos u + h(t)|u'|^\lambda \operatorname{sgn} u'(t). \qquad \square$$

The following lemma obviously holds.

Lemma 4.2. *Let*

$$p(t) \ge 0 \ \text{ for a.e. } \ t \in [a, b], \quad p(t) \not\equiv 0 \ \text{ on } [a, b]. \tag{4.10}$$

Then $\alpha(t) \stackrel{\text{def}}{=} -\frac{\pi}{2}$, $\beta(t) \stackrel{\text{def}}{=} \frac{\pi}{2}$ *for* $t \in [a, b]$ *are the well-ordered lower and upper functions of equation* (4.3).

The following proposition concerning the boundary conditions

$$u'(a) = 0, \quad u(a) = u(b) \tag{4.11}$$

follows from Theorems 3.2, 3.3 and Lemmas 4.1, 4.2.

Proposition 4.1. *Let* $\lambda \ge 1$ *and* $p, g, h \in L^1([a, b]; \mathbb{R})$ *be such that* (4.10) *holds and*

$$\left.\begin{aligned} &[h]_+ \in L^{\frac{1}{2-\lambda}}([a, b]; \mathbb{R}_+), && \text{if } \lambda \in \,]1, 2[\,, \\ &[h]_+ \in L^\infty([a, b]; \mathbb{R}_+), && \text{if } \lambda = 2, \\ &[h(t)]_+ \equiv 0 \ \text{ on } [a, b], && \text{if } \lambda > 2. \end{aligned}\right\} \tag{4.12}$$

Then problem (4.3), (4.11) *has a maximal solution* u^* *and a minimal solution* u_* *in the segment* $[-\frac{\pi}{2}, \frac{\pi}{2}]$, *and* $-\frac{\pi}{2} < u_*(t) \le u^*(t) < \frac{\pi}{2}$ *for* $t \in [a, b]$.

For the Dirichlet problem, from Theorems 1.1, 3.4 and Lemmas 4.1, 4.2, we get

Proposition 4.2. *Let* $\lambda \ge 1$ *and* $p, g, h \in L^1([a, b]; \mathbb{R})$ *be such that* (4.10) *and* (4.12) *are fulfilled. Then, for any* $c_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *and* $c_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, *problem* (4.3), (1.4) *has a maximal solution* u^* *and a minimal solution* u_* *in the segment* $[-\frac{\pi}{2}, \frac{\pi}{2}]$, *and* $-\frac{\pi}{2} < u_*(t) \le u^*(t) < \frac{\pi}{2}$ *for* $t \in \,]a, b[\,$.

For the boundary conditions

$$u'(a) = 0, \quad u(b) = c, \tag{4.13}$$

Theorems 1.2, 3.5 and Lemmas 4.1, 4.2 yield

Proposition 4.3. *Let* $\lambda \ge 1$ *and* $p, g, h \in L^1([a, b]; \mathbb{R})$ *be such that* (4.10) *and* (4.12) *are fulfilled. Then, for any* $c \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, *problem* (4.3), (4.13) *has a maximal solution* u^* *and a minimal solution* u_* *in the segment* $[-\frac{\pi}{2}, \frac{\pi}{2}]$, *and* $-\frac{\pi}{2} < u_*(t) \le u^*(t) < \frac{\pi}{2}$ *for* $t \in [a, b[\,$.

If the coefficient g does not change its sign, Proposition 4.1 can be refined.

Proposition 4.4. *Let $i \in \{1,2\}$ and $p, g, h \in L^1([a,b];\mathbb{R})$ be such that (4.10) and (4.12) hold and*

$$(-1)^i g(t) \geq 0 \ \ \textit{for a.e.} \ \ t \in [a,b]. \tag{4.14}$$

Then problem (4.3), (4.11) has a unique solution u satisfying

$$-\frac{\pi}{2} \leq u(t) \leq \frac{\pi}{2} \ \ \textit{for} \ \ t \in [a,b] \tag{4.15}$$

and, moreover,

$$0 \leq (-1)^{i+1} u(t) < \frac{\pi}{2} \ \ \textit{for} \ \ t \in [a,b]. \tag{4.16}$$

Proof. Assume that $i = 1$. Let u be a solution to problem (4.3), (4.11) satisfying (4.15). Proposition 4.1 yields $-\frac{\pi}{2} < u(t) < \frac{\pi}{2}$ for $t \in [a,b]$. We show that

$$u(t) \geq 0 \ \ \text{for} \ \ t \in [a,b]. \tag{4.17}$$

Suppose on the contrary that (4.17) does not hold. Then, in view of (4.11), there exists $t_0 \in [a,b[$ such that

$$u(t_0) < 0, \quad u'(t_0) = 0. \tag{4.18}$$

Put

$$b_0 = \sup\big\{t \in [t_0,b] : \ u(s) < 0 \text{ for } s \in [t_0,t]\big\}.$$

It is clear that

$$u(t) < 0 \ \ \text{for} \ \ t \in [t_0,b_0[\,, \quad u(b_0) = 0 \ \ \text{if} \ \ b_0 < b. \tag{4.19}$$

By virtue of (4.10) and (4.14), it follows from (4.3) that

$$u''(t) = p(t)\sin u(t) + g(t)\cos u(t) + h(t)|u'(t)|^\lambda \operatorname{sgn} u'(t) \leq h(t)|u'(t)|^{\lambda-1}u'(t) \ \ \text{for a.e.} \ \ t \in [t_0,b_0]$$

(we put $|u'(t)|^{\lambda-1} = 1$ if $\lambda = 1$), which yields

$$\Big(u'(t)\,\mathrm{e}^{\int_{t_0}^{t} h(s)|u'(s)|^{\lambda-1}\,\mathrm{d}s}\Big)' \leq 0 \ \ \text{for a.e.} \ \ t \in [t_0,b_0].$$

This inequality, together with the second condition in (4.18), results in

$$u'(t) \leq 0 \ \ \text{for} \ \ t \in [t_0,b_0], \tag{4.20}$$

and therefore, using (4.19), we get $b_0 = b$ and $u(b) < 0$. Consequently, the second condition in (4.11) implies $u(a) < 0$.

If $t_0 > a$, we put $a_0 = \sup\{t \in [a,t_0] : \ u(s) < 0 \text{ for } s \in [a,t]\}$. It is clear that

$$u(t) < 0 \ \ \text{for} \ \ t \in [a,a_0[\,, \quad u(a_0) = 0 \ \ \text{if} \ \ a_0 < t_0. \tag{4.21}$$

In a similar manner as above, we obtain

$$\Big(u'(t)\,\mathrm{e}^{\int_{a}^{t} h(s)|u'(s)|^{\lambda-1}\,\mathrm{d}s}\Big)' \leq 0 \ \ \text{for a.e.} \ \ t \in [a,a_0]$$

(we put $|u'(s)|^{\lambda-1} = 1$ if $\lambda = 1$) which, in view of the first condition in (4.11), yields

$$u'(t) \leq 0 \ \ \text{for} \ \ t \in [a,a_0]. \tag{4.22}$$

Therefore, using (4.21), we get $a_0 = t_0$, and thus, it follows from (4.20) and (4.22) that

$$u'(t) \leq 0 \ \ \text{for} \ \ t \in [a,b], \tag{4.23}$$

since we have proved that $b_0 = b$. If $t_0 = a$, inequality (4.23) follows immediately from (4.20).

Now, the second condition in (4.11) and (4.23) lead to $u(t) = u(t_0)$ for $t \in [a, b]$. Since $u(t_0) \in]-\frac{\pi}{2}, 0[$, in view of (4.10) and (4.14), from (4.3) we get $0 = p(t)\sin u(t_0) + g(t)\cos u(t_0) \le p(t)\sin u(t_0) \le 0$ for a.e. $t \in [a, b]$. Consequently, $p(t) = 0$ for a.e. $t \in [a, b]$, which contradicts (4.10). The obtained contradiction proves that for any solution u to problem (4.3), (4.11) satisfying (4.15), condition (4.16) holds.

We now show that problem (4.3), (4.11) has a unique solution u satisfying (4.15). According to Proposition 4.1, there are a maximal solution u^* and a minimal solution u_* of problem (4.3), (4.11) in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$. It follows from the above-proved fact that

$$0 \le u_*(t) \le u^*(t) < \frac{\pi}{2} \quad \text{for } t \in [a, b]. \tag{4.24}$$

Suppose that

$$\max\big\{u^*(t) - u_*(t) :\ t \in [a, b]\big\} > 0. \tag{4.25}$$

Since, for any $t^* \in [a, b]$ and $c_1, c_2 \in \mathbb{R}$, the Cauchy problem

$$u'' = p(t)\sin u + g(t)\cos u + h(t)|u'|^{\lambda}\operatorname{sgn} u'(t); \quad u(t^*) = c_1, \ \ u'(t^*) = c_2$$

is uniquely solvable, in view of (4.24) and (4.25), we get

$$0 \le u_*(t) < u^*(t) < \frac{\pi}{2} \quad \text{for } t \in [a, b]. \tag{4.26}$$

Put $w(t) \overset{\text{def}}{=} u^*(t) - u_*(t)$ for $t \in [a, b]$ and

$$\gamma(x, y) \overset{\text{def}}{=} \begin{cases} \dfrac{|x|^{\lambda}\operatorname{sgn} x - |y|^{\lambda}\operatorname{sgn} y}{x - y} & \text{for } x, y \in \mathbb{R}, \ \ x \ne y, \\ \lambda |x|^{\lambda-1} & \text{for } x, y \in \mathbb{R}, \ \ x = y \end{cases}$$

(we put $|x|^{\lambda-1} = 1$ if $\lambda = 1$). One can show that the function γ is continuous on $\mathbb{R}^2$. Then, in view of (4.10), (4.14), and (4.26), equation (4.3) yields

$$\begin{aligned} w''(t) &= p(t)\big[\sin u^*(t) - \sin u_*(t)\big] + g(t)\big[\cos u^*(t) - \cos u_*(t)\big] \\ &\quad + h(t)\big[|u^{*\prime}(t)|^{\lambda}\operatorname{sgn} u^{*\prime}(t) - |u_*'(t)|^{\lambda}\operatorname{sgn} u_*'(t)\big] \ge h(t)\gamma(u^{*\prime}(t), u_*'(t))w'(t) \ \text{ for a.e. } t \in [a, b], \end{aligned}$$

which leads to

$$\Big(w'(t)\, \mathrm{e}^{\int_a^t h(s)\gamma(u^{*\prime}(s), u_*'(s))\, \mathrm{d}s}\Big)' \le 0 \ \text{ for a.e. } t \in [a, b]. \tag{4.27}$$

Moreover, using (4.11), we get

$$w'(a) = 0, \quad w(a) = w(b), \tag{4.28}$$

and thus it follows from (4.27) that $w'(t) \ge 0$ for $t \in [a, b]$. Therefore, by virtue of the second condition in (4.28), the latter inequality guarantees that $w(t) = w(a)$ for $t \in [a, b]$. Hence, taking into account (4.10), (4.14), and (4.26), from (4.3) we arrive at

$$\begin{aligned} 0 = p(t)\big[\sin u^*(t) - \sin u_*(t)\big] &+ g(t)\big[\cos u^*(t) - \cos u_*(t)\big] \\ &\ge p(t)\big[\sin u^*(t) - \sin u_*(t)\big] \ge 0 \ \text{ for a.e. } t \in [a, b]. \end{aligned}$$

Consequently, $p(t) = 0$ for a.e. $t \in [a, b]$ (see (4.26)), which contradicts (4.10). The obtained contradiction proves that $u^*(t) = u_*(t)$ for $t \in [a, b]$, and therefore, problem (4.3), (4.11) has a unique solution u satisfying (4.15).

For $i = 2$, the conclusions of the proposition follows from the above-proved case $i = 1$ and the fact that u is a solution to problem (4.3), (4.11) if and only if $-u$ is a solution to the problem

$$u'' = p(t)\sin u - g(t)\cos u + h(t)|u'|^{\lambda}\operatorname{sgn} u'(t); \quad u(a') = 0, \ \ u(a) = u(b). \qquad \square$$

4.1 On the Whitney problem

As a possible use of Propositions 4.1, 4.2, 4.3, and 4.4, we provide three modifications of the Whitney problem, which is formulated on p. 104.

Question 4.1. Is there any initial position θ_0 of the rod such that if the motion of the rod starts at station A (i.e., at time $t = 0$) from the position θ_0 with zero angular velocity, then the rod is again at station B (i.e., at time $t = T$) in the position θ_0, without the rod touching the floor during the motion?

Propositions 4.1 and 4.4 yield

Corollary 4.1. *The problem*

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell}\cos\theta - \frac{\mathrm{g}}{\ell}\sin\theta = 0; \quad \dot{\theta}(0) = 0, \;\; \theta(0) = \theta(T) \tag{4.29}$$

has a maximal solution θ^* and a minimal solution θ_* in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $-\frac{\pi}{2} < \theta_*(t) \leq \theta^*(t) < \frac{\pi}{2}$ for $t \in [0, T]$.

Moreover, if

$$\textit{either } \ddot{\varphi}(t) \geq 0 \;\textit{ for a.e. }\; t \in [0, T], \quad \textit{or } \ddot{\varphi}(t) \leq 0 \;\textit{ for a.e. }\; t \in [0, T],$$

then problem (4.29) has a unique solution θ satisfying (4.2).

Answer 4.1. Yes; $\theta_0 = \theta(0)$, where θ is a solution to problem (4.29).

Question 4.2. Let the initial position θ_0 and the terminal position θ_T of the rod be given. Is there any initial angular velocity ω_0 of the rod such that if the motion of the rod starts at station A (i.e., at time $t = 0$) from the position θ_0 with the velocity ω_0, then the rod is at station B (i.e., at time $t = T$) in the position θ_T, without the rod touching the floor during the motion?

Proposition 4.2 implies

Corollary 4.2. *For any $\theta_0, \theta_T \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the problem*

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell}\cos\theta - \frac{\mathrm{g}}{\ell}\sin\theta = 0; \quad \theta(0) = \theta_0, \;\; \theta(T) = \theta_T \tag{4.30}$$

has a maximal solution θ^* and a minimal solution θ_* in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $-\frac{\pi}{2} < \theta_*(t) \leq \theta^*(t) < \frac{\pi}{2}$ for $t \in]0, T[$.

Answer 4.2. Yes; $\omega_0 = \dot{\theta}(0)$, where θ is a solution to problem (4.30).

Question 4.3. Let the terminal position θ_T of the rod be given. Is there any initial position θ_0 of the rod such that if the motion of the rod starts at station A (i.e., at time $t = 0$) from the position θ_0 with zero angular velocity, then the rod is at station B (i.e., at time $t = T$) in the position θ_T, without the rod touching the floor during the motion?

Proposition 4.2 implies

Corollary 4.3. *For any $\theta_T \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the problem*

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell}\cos\theta - \frac{\mathrm{g}}{\ell}\sin\theta = 0; \quad \dot{\theta}(0) = 0, \;\; \theta(T) = \theta_T \tag{4.31}$$

has a maximal solution θ^* and a minimal solution θ_* in the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and $-\frac{\pi}{2} < \theta_*(t) \leq \theta^*(t) < \frac{\pi}{2}$ for $t \in [0, T[$.

Answer 4.3. Yes; $\theta_0 = \theta(0)$, where θ is a solution to problem (4.31).

Observe that using Corollary 4.3, one can easily prove

Corollary 4.4. *Let $\theta_{\max} := \theta^*(0)$, where θ^* is a maximal solution to the problem*

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell} \cos\theta - \frac{\mathrm{g}}{\ell} \sin\theta = 0; \quad \dot{\theta}(0) = 0, \ \ \theta(T) = \frac{\pi}{2}\,.$$

Then, for any $\theta_0 \in \left]\theta_{\max}, \frac{\pi}{2}\right[$, the initial value problem

$$\ddot{\theta} + \frac{\ddot{\varphi}(t)}{\ell} \cos\theta - \frac{\mathrm{g}}{\ell} \sin\theta = 0; \quad \theta(0) = \theta_0, \ \ \dot{\theta}(0) = 0$$

has a unique solution θ and this solution satisfies $\theta(t) = \frac{\pi}{2}$ for some $t \in \left]0, T\right[$.

It follows from Corollary 4.4 that if the motion of the rod starts at station A (i.e., at time $t = 0$) from the position $\theta_0 \in \left]\theta_{\max}, \frac{\pi}{2}\right[$ with zero angular velocity, then the rod falls to the floor before the train reaches station B.

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