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SOLUTION OF THE CAUCHY PROBLEM FOR IMPULSIVE LINEAR SET-VALUED DIFFERENTIAL EQUATIONS WITH A CONFORMABLE FRACTIONAL-FRACTAL DERIVATIVE

Dedicated to the blessed memory of Professor M. O. Perestyuk

Abstract. The article presents a conformal fractional-fractal derivative for set-valued mappings, unifying conformal fractional and fractal derivatives. Key properties are derived. Analytical solutions are obtained for linear set-valued Cauchy problems, both without impulses and with impulsive effects, involving this derivative.

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1 Introduction

The theory of differential equations with impulse action began with the article by V. D. Mil'man and A. D. Myshkis [17] in 1960. Impulse differential equations, which describe phenomena involving impulse actions, provide a natural framework for modeling the evolution of many real-world problems. The basis of this theory was the fundamental results of A. M. Samoylenko and M. O. Perestyuk, as presented in their monographs [26, 27]. Later, many scientists studied this theory, contributing to its rapid development (see [2–5, 13, 19, 23, 25] and the references therein).

As is known, an impulse system consists of three components: a continuous-time differential equation, which governs the state of the system between impulses; an impulse equation that models the impulse action; and the conditions under which the impulse action occurs. It is evident that the development of the theory of differential equations with fractional derivatives, set-valued differential equations, and fuzzy differential equations has led to corresponding results in the theory of impulsive systems (see [1, 5, 6, 9, 14, 15, 19–23, 28, 29, 31] and the references therein).

The article is devoted to the linear set-valued Cauchy problem with the conformable fractional-fractal derivative and is organized as follows. Section 2 of this article introduces the notion of a conformable fractional-fractal derivative for a set-valued mappings, which extends both the conformable fractional derivative and the fractal derivative. Its fundamental properties are established and proved. Section 3 addresses the Cauchy problem for a linear set-valued differential equation involving the conformable fractional-fractal derivative and derives an explicit analytical solution. Finally, Section 4 examines the Cauchy problem for an impulsive linear set-valued differential equation with the conformable fractional-fractal derivative and also presents an analytical solution.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^n$ be the n-dimensional Euclidean space ($n \geq 2$). Denote by $conv(\mathbb{R}^n)$ the set of nonempty compact and convex subsets of $\mathbb{R}^n$ with the Hausdorff metric

$$h(X,Y) = \max\Big\{ \sup_{x\in X} \inf_{y\in Y} \|x-y\|,\ \sup_{y\in Y} \inf_{x\in X} \|x-y\| \Big\},$$

where $X, Y \in conv(\mathbb{R}^n)$, ($\|\cdot\|$ denotes the Euclidean norm).

In addition to the usual set-theoretic operations, the following operations in the space $conv(\mathbb{R}^n)$ are introduced: the sum of the sets, the product of the scalar on the set and the operation of the product of the matrix on the set:

$$X+Y = \bigcup_{x\in X,\ y\in Y} \{x+y\}, \quad \lambda X = \bigcup_{x\in X} \{\lambda x\}, \quad AX = \bigcup_{x\in X} \{Ax\},$$

where $X, Y \in conv(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$.

Lemma 2.1 ([24]). *The following properties hold:*

(1) $(conv(\mathbb{R}^n), h)$ *is a complete metric space;*

(2) $X+Y \in conv(\mathbb{R}^n)$, $X+Y = Y+X$, $(X+Y)+Z = X+(Y+Z)$;

(3) $\alpha X \in conv(\mathbb{R}^n)$;

(4) $\alpha(\beta X) = (\alpha\beta)X$, $\alpha(X+Y) = \alpha X + \alpha Y$, $(\alpha+\beta)X \subseteq \alpha X + \beta X$;

(5) $AX \in conv(\mathbb{R}^n)$;

(6) $A(X+Y) = AX + AY$, $(A+B)X \subseteq AX + BX$;

(7) $h(X+Z, Y+Z) = h(X,Y)$;

(8) $h(\alpha X, \alpha Y) = |\alpha| h(X,Y)$ *for all* $X, Y, Z \in conv(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{R}$ *and* $A, B \in \mathbb{R}^{n\times n}$.

However, $conv(\mathbb{R}^n)$ is not a linear space, since it does not contain inverse elements for addition, and hence the difference is not well-defined. That is, if $X \in conv(\mathbb{R}^n)$ and $X \neq \{\mathbf{x}\}$, then there doesn't exist $(-X)$ such that $X + (-X) \neq \{\mathbf{0}\}$, where $\mathbf{0} = (0, \ldots, 0)^T$ is the zero vector. As a consequence, alternative formulations for difference have been suggested in [12,19,22–24]. One of these alternatives is the Hukuhara difference [10].

Definition 2.1 ([10])**.** Let $X, Y \in conv(\mathbb{R}^n)$. A set $Z \in conv(\mathbb{R}^n)$ such that $X = Y + Z$ is called a **Hukuhara difference (H-difference)** of the sets X and Y and is denoted by $X \overset{H}{-} Y$.

In this case, $X \overset{H}{-} X = \{\mathbf{0}\}$ and $(X + Y) \overset{H}{-} Y = X$ for any $X, Y \in conv(\mathbb{R}^n)$, but obviously, $X \overset{H}{-} Y \neq X + (-1)Y$. The properties of this difference are studied in detail in [19,22–24].

Let $X : [0, T] \to conv(\mathbb{R}^n)$ be a set-valued mapping; $k : [0, T] \times (0, 1] \to \mathbb{R}_+$ be a continuous function such that $k(t, \alpha) > 0$ for all $(t, \alpha) \in (0, T] \times (0, 1]$, and $k(t, 1) = 1$ for all $t \in [0, T]$; $m : [0, T] \times (0, 1] \to \mathbb{R}_+$ be a continuously differentiable function such that $m(t, \beta) > 0$ and $\frac{\partial m(t,\beta)}{\partial t} > 0$ for all $(t, \beta) \in (0, T] \times (0, 1]$, and $\frac{\partial m(t,1)}{\partial t} \equiv 1$ for all $t \in [0, T]$.

Definition 2.2. Let $t \in (0, T)$ and $\alpha, \beta \in (0, 1]$. If the Hukuhara differences $X(t + \varepsilon k(t, \alpha)) \overset{H}{-} X(t)$ and $X(t) \overset{H}{-} X(t - \varepsilon k(t, \alpha))$ exist for all sufficiently small $\varepsilon > 0$ and there exists $Z^\alpha_\beta \in conv(\mathbb{R}^n)$ such that the equality

$$\lim_{\varepsilon \downarrow 0} \big(m(t + \varepsilon, \beta) - m(t, \beta)\big)^{-1} \big(X(t + \varepsilon k(t, \alpha)) \overset{H}{-} X(t)\big)$$
$$= \lim_{\varepsilon \downarrow 0} \big(m(t, \beta) - m(t - \varepsilon, \beta)\big)^{-1} \big(X(t) \overset{H}{-} X(t - \varepsilon k(t, \alpha))\big) = Z^\alpha_\beta \quad (2.1)$$

holds, then we say that the set-valued mapping $X(\cdot)$ has a **conformable fractional-fractal derivative of order** (α, β) at the point $t \in (0, T)$ and $D^\alpha_\beta X(t) = Z^\alpha_\beta$.

If $D^\alpha_\beta X(t)$ exists for all $t \in (0, T)$ and $\lim_{t \downarrow 0} D^\alpha_\beta X(t)$ and $\lim_{t \uparrow T} D^\alpha_\beta X(t)$ exist, then we assume that $D^\alpha_\beta X(0) = \lim_{t \downarrow 0} D^\alpha_\beta X(t)$ and $D^\alpha_\beta X(T) = \lim_{t \uparrow T} D^\alpha_\beta X(t)$.

Definition 2.3. If the conformable fractional-fractal derivative of order (α, β) $D^\alpha_\beta X(t)$ exists for all $t \in [0, T]$, then we say that the set-valued mapping $X(\cdot)$ is (α, β)**-differentiable** on $[0, T]$.

Remark 2.1. If $k(t, \alpha) = t^{1-\alpha}$ and $m(t, \beta) = t$, then equality (2.1) has the form

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \big(X(t + \varepsilon t^{1-\alpha}) \overset{H}{-} X(t)\big) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \big(X(t) \overset{H}{-} X(t - \varepsilon t^{1-\alpha})\big) = Z^\alpha,$$

and $D^\alpha_\beta X(t)$ is **a conformable fractional derivative of order α** $D^\alpha X(t)$ at the point $t \in (0, T)$, that was considered in [11,16,18,30].

Remark 2.2. If $k(t, \alpha) \equiv 1$ and $m(t, \beta) = t^\beta$, then equality (2.1) has the form

$$\lim_{\varepsilon \downarrow 0} \big((t + \varepsilon)^\beta - t^\beta\big)^{-1} \big(X(t + \varepsilon) \overset{H}{-} X(t)\big) = \lim_{\varepsilon \downarrow 0} \big(t^\beta - (t - \varepsilon)^\beta\big)^{-1} \big(X(t) \overset{H}{-} X(t - \varepsilon)\big) = Z_\beta,$$

and $D^\alpha_\beta X(t)$ is **a fractal derivative of order** β $D_\beta X(t)$ at the point $t \in (0, T)$, that generalizes the Chen–Hausdorff fractal derivative [7,8] to the case of set-valued mappings.

Next, we provide some properties of the conformable fractional-fractal derivative of order (α, β).

Lemma 2.2. *If the set-valued mapping $X(\cdot)$ is (α, β)-differentiable at the point $t \in (0, T)$, then $D^\alpha_\beta X(t) = (m'_t(t, \beta))^{-1} k(t, \alpha) D_H X(t)$, where $D_H X(t)$ is the Hukuhara derivative* [10].

Proof. If the set-valued mapping $X(\cdot)$ is (α, β)-differentiable at the point $t \in (0, T)$, then we have

$$D^\alpha_\beta X(t) = \lim_{\varepsilon \downarrow 0} \big(m(t + \varepsilon, \beta) - m(t, \beta)\big)^{-1} \big(X(t + \varepsilon k(t, \alpha)) \overset{H}{-} X(t)\big)$$
$$= \lim_{\varepsilon \downarrow 0} \big(m(t + \varepsilon, \beta) - m(t, \beta)\big)^{-1} \varepsilon \cdot \varepsilon^{-1} \big(X(t + \varepsilon k(t, \alpha)) \overset{H}{-} X(t)\big)$$

$$= \lim_{\varepsilon\downarrow 0} \big(m(t+\varepsilon,\beta) - m(t,\beta)\big)^{-1}\varepsilon \cdot \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\big(X(t+\varepsilon k(t,\alpha)) \overset{H}{-} X(t)\big)$$
$$= \frac{1}{\lim\limits_{\varepsilon\downarrow 0} \frac{m(t+\varepsilon,\beta)-m(t,\beta)}{\varepsilon}} \cdot \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\big(X(t+\varepsilon k(t,\alpha)) \overset{H}{-} X(t)\big).$$

As $m(t,\beta)$ is differentiable with respect to t, then

$$\frac{1}{\lim\limits_{\varepsilon\downarrow 0} \frac{m(t+\varepsilon,\beta)-m(t,\beta)}{\varepsilon}} = (m_t'(t,\beta))^{-1}.$$

Let $h = \varepsilon k(t,\alpha)$. It is obvious that $\lim\limits_{\varepsilon\downarrow 0} \varepsilon k(t,\alpha) = 0_+$ for all $t \in (0,T)$. Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\big(X(t+\varepsilon k(t,\alpha)) \overset{H}{-} X(t)\big) = \lim_{h\downarrow 0} k(t,\alpha)h^{-1}\big(X(t+h) \overset{H}{-} X(t)\big) = k(t,\alpha)D_H X(t),$$

i.e., $D^\alpha_\beta X(t) = (m_t'(t,\beta))^{-1}k(t,\alpha)D_H X(t)$.

Similarly,

$$D^\alpha_\beta X(t) = \lim_{\varepsilon\downarrow 0} \big(m(t,\beta) - m(t-\varepsilon,\beta)\big)^{-1}\big(X(t) \overset{H}{-} X(t-\varepsilon k(t,\alpha))\big)$$
$$= \frac{1}{\lim\limits_{\varepsilon\downarrow 0} \frac{m(t,\beta)-m(t-\varepsilon,\beta)}{\varepsilon}} \cdot \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\big(X(t) \overset{H}{-} X(t-\varepsilon k(t,\alpha))\big)$$
$$= (m_t'(t,\beta))^{-1}k(t,\alpha)D_H X(t). \qquad \square$$

Remark 2.3. From Lemma 2.2 we conclude that the necessary and sufficient condition for the existence of a conformable fractional-fractal derivative $D^\alpha_\beta X(t)$ of order (α,β) for the set-valued mapping $X(\,\cdot\,)$ is the existence of the Hukuhara derivative $D_H X(t)$.

Remark 2.4. From Lemma 2.2 and Remark 2.3, we have that if a mapping $X(\,\cdot\,)$ is (α,β)-differentiable, then it is continuous.

Lemma 2.3. *If the set-valued mapping $X(t) \equiv X$ for all $t \in [0,T]$, then $D^\alpha_\beta X(t) = \{\mathbf{0}\}$.*

Proof. By Lemma 2.2, we have $D^\alpha_\beta X(t) = (m_t'(t,\beta))^{-1}k(t,\alpha)D_H X(t)$. As $X(t) \equiv X$ for all $t \in [0,T]$ and $D_H X = \{\mathbf{0}\}$, then $D^\alpha_\beta X(t) = (m_t'(t,\beta))^{-1}k(t,\alpha)\{\mathbf{0}\} = \{\mathbf{0}\}$. $\square$

Lemma 2.4. *If the set-valued mappings $X(\,\cdot\,)$ and $Y(\,\cdot\,)$ are (α,β)-differentiable at the point $t \in (0,T)$, then $D^\alpha_\beta(AX(t) + BY(t)) = AD^\alpha_\beta X(t) + BD^\alpha_\beta Y(t)$, where $A, B \in \mathbb{R}^{n\times n}$.*

Proof. By Lemma 2.2, we have

$$D^\alpha_\beta X(t) = (m_t'(t,\beta))^{-1}k(t,\alpha)D_H X(t) \;\text{ and }\; D^\alpha_\beta Y(t) = (m_t'(t,\beta))^{-1}k(t,\alpha)D_H Y(t).$$

Then

$$AD^\alpha_\beta X(t) + BD^\alpha_\beta Y(t) = A(m_t'(t,\beta))^{-1}k(t,\alpha)D_H X(t) + B(m_t'(t,\beta))^{-1}k(t,\alpha)D_H Y(t).$$

Likewise,

$$D^\alpha_\beta(AX(t) + BY(t)) = (m_t'(t,\beta))^{-1}k(t,\alpha)D_H(AX(t) + BY(t))$$
$$= A(m_t'(t,\beta))^{-1}k(t,\alpha)D_H X(t) + B(m_t'(t,\beta))^{-1}k(t,\alpha)D_H Y(t). \qquad \square$$

Definition 2.4. The **conformable fractional-fractal integral** of set-valued mapping $X(\,\cdot\,)$ is defined by $I^\alpha_\beta X(t) = \int\limits_0^t m_s'(s,\beta)k^{-1}(s,\alpha)X(s)\,ds$, $t \geq 0$, where the integral on the right-hand side is understood in the sense of Hukuhara integral [10].

Lemma 2.5. *If the set-valued mapping $X(\cdot)$ is continuous on $[0,T]$, then $D^\alpha_\beta I^\alpha_\beta X(t)=X(t)$, $t\in(0,T)$.*

Proof. If the set-valued mapping $X(\cdot)$ is continuous on $[0,T]$, then $I^\alpha_\beta X(t)=\int_0^t m'_s(s,\beta)k^{-1}(s,\alpha)X(s)\,ds$ exists for all $t\in[0,T]$. By [10], the set-valued mapping $I^\alpha_\beta X(t)$ is Hukuhara differentiable and $D^\alpha_\beta I^\alpha_\beta X(t)=(m'_t(t,\beta))^{-1}k(t,\alpha)D_H I^\alpha_\beta X(t)$. Then

$$\begin{aligned} D^\alpha_\beta I^\alpha_\beta X(t) &= (m'_t(t,\beta))^{-1}k(t,\alpha)D_H\left(\int_0^t m'_s(s,\beta)k^{-1}(s,\alpha)X(s)\,ds\right) \\ &= (m'_t(t,\beta))^{-1}k(t,\alpha)m'_t(t,\beta)k^{-1}(t,\alpha)X(t)=X(t). \end{aligned}$$ □

Lemma 2.6. *If the set-valued mapping $X(\cdot)$ is (α,β)-differentiable on $[0,T]$, then $I^\alpha_\beta D^\alpha_\beta X(t)=X(t)\overset{H}{-}X(0)$.*

Proof. If the set-valued mapping $X(\cdot)$ is (α,β)-differentiable on $[0,T]$, then

$$D^\alpha_\beta X(t)=(m'_t(t,\beta))^{-1}k(t,\alpha)D_H X(t).$$

Hence,

$$I^\alpha_\beta D^\alpha_\beta X(t)=\int_0^t m'_s(s,\beta)k^{-1}(s,\alpha)(m'_s(s,\beta))^{-1}k(s,\alpha)D_H X(s)\,ds=\int_0^t D_H X(s)\,ds=X(t)\overset{H}{-}X(0).$$ □

Remark 2.5. Obviously, if $\alpha=1$ and $m'_t(t,1)\equiv 1$, then $D^1_1X(t)$ coincides with the Hukuhara derivative and $I^1_1X(t)$ coincides with the Hukuhara integral.

This also holds for the derivatives considered in Remarks 2.1 and 2.2. As the corresponding integrals have the form

$$I^\alpha X(t)=\int_0^t s^{\alpha-1}X(s)\,ds \quad\text{and}\quad I_\beta X(t)=\int_0^t \beta s^{\beta-1}X(s)\,ds,$$

then

$$I^1X(t)=I_1X(t)=\int_0^t X(s)\,ds.$$

3 Linear set-valued differential equation with conformable fractional-fractal derivative

Consider the Cauchy problem for a linear set-valued differential equation of order (α,β)

$$D^\alpha_\beta X(t)=AX(t),\quad X(0)=X_0, \tag{3.1}$$

where $\alpha,\beta\in(0,1]$, $X:[0,T]\to conv(\mathbb{R}^n)$ is an unknown set-valued mapping, $A\in\mathbb{R}^{n\times n}$ is a nondegenerate matrix, $X_0\in conv(\mathbb{R}^n)$.

Definition 3.1. A set-valued mapping $X_{\alpha\beta}:[0,T]\to conv(\mathbb{R}^n)$ is called a solution of the Cauchy problem (3.1) if it is continuous and satisfies differential equation (3.1) for all $t\in[0,T]$ and $X(0)=X_0$.

Theorem 3.1. *The Cauchy problem (3.1) has the following solution:*

$$X_{\alpha\beta}(t)=X_0+\sum_{i=1}^{\infty}\left\{\frac{\left[\int_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^i}{i!}A^iX_0\right\}, \tag{3.2}$$

where $t\in[0,T]$.

Proof. First, we establish some properties of the set-valued mapping (3.2).

Let $\delta = h(X_0, \mathbf{0})$, $\eta = \|A\|$, $\gamma = \max\limits_{x,y\in X_0} \|x-y\|$ and let $\mathbf{c}$ be such that $X_0 \subseteq B_{\frac{\gamma}{2}}(\mathbf{c}) = \{x \in \mathbb{R}^n : \|x-\mathbf{c}\| \le \frac{\gamma}{2}\}$. Let also

$$X_i(t,\alpha,\beta) = \frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!} A^i X_0, \quad X^j(t,\alpha,\beta) = \sum_{i=0}^{j} X_i(t,\alpha,\beta).$$

As $m'_t(t,\beta) > 0$ and $k(t,\alpha) > 0$ for all $t \in [0,T]$, $\alpha, \beta \in (0,1]$, then

$$\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds < \int\limits_0^T m'_s(s,\beta)k^{-1}(s,\alpha)\,ds = \chi(T,\alpha,\beta)$$

for all $t \in [0,T]$ and

$$\sum_{i=0}^{\infty} \frac{\chi(T,\alpha,\beta)^i}{i!}\,\eta^i\delta = e^{\chi(T,\alpha,\beta)\eta}\delta.$$

Since for any $p > q$ $(p,q \in \mathbb{N})$ and for all $t \in [0,T]$,

$$\begin{aligned} h(X^p(t,\alpha,\beta), X^q(t,\alpha,\beta)) &\le h\left(\sum_{i=q+1}^{p}\left\{\frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!} A^i X_0\right\}, \{\mathbf{0}\}\right) \\ &\le \delta \sum_{i=q+1}^{p}\left\{\frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!}\,\eta^i\right\} \\ &\le \delta \sum_{i=q+1}^{p}\left\{\frac{\chi(T,\alpha,\beta)^i}{i!}\,\eta^i\right\}, \end{aligned}$$

for any $\varepsilon > 0$ one can find $N(\varepsilon) > 0$ such that for all $p > q > N(\varepsilon)$ inequality $h(X^p(t,\alpha,\beta), X^q(t,\alpha,\beta)) < \varepsilon$ is satisfied, i.e., the sequence $\{X^k(t,\alpha,\beta)\}_{k=0}^{\infty}$ converges uniformly to $X_{\alpha\beta}(t)$.

Also, as $X_0 \subseteq B_{\frac{\gamma}{2}}(\mathbf{c}) = \mathbf{c} + \frac{\gamma}{2}\,B_1(\mathbf{0})$, and for all $i = 1, 2, \ldots,$

$$X_i(t,\alpha,\beta) \subseteq \frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!} A^i\mathbf{c} + \frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!}\,\|A^i\|\,\frac{\gamma}{2}\,B_1(\mathbf{0}),$$

then

$$\begin{aligned} X_{\alpha\beta}(t) &\subseteq \sum_{i=0}^{\infty} \frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!} A^i\mathbf{c} + \sum_{i=0}^{\infty} \frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!}\,\|A^i\|\,\frac{\gamma}{2}\,B_1(\mathbf{0}) \\ &\subseteq e^{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]A}\,\mathbf{c} + \frac{\gamma}{2}\,e^{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]\|A\|}\,B_1(\mathbf{0}). \end{aligned}$$

Let us prove that $X_{\alpha\beta}(\,\cdot\,)$ is a solution of the Cauchy problem (3.1) by directly substituting the set-valued mapping $X_{\alpha\beta}(\,\cdot\,)$ into the differential equation (3.1) and checking that the identity $D^{\alpha}_{\beta} X_{\alpha\beta}(t) = AX_{\alpha\beta}(t)$ is satisfied.

By Lemma 2.2, we have $D^{\alpha}_{\beta} X_{\alpha\beta}(t) = (m'_t(t,\beta))^{-1}k(t,\alpha)D_H X_{\alpha\beta}(t)$. Since $D_H X_0 = \{\mathbf{0}\}$ and

$$D_H\left(\frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^i}{i!} A^i X_0\right) = m'_t(t,\beta)k^{-1}(t,\alpha)\,\frac{\Big[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\Big]^{i-1}}{(i-1)!} A^i X_0$$

for all $i = 1, 2, \ldots$, and

$$\sum_{i=1}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i-1}}{(i-1)!}A^iX_0\right\}$$

uniformly converges, we have

$$D_H X_{\alpha\beta}(t) = m'_t(t,\beta)k^{-1}(t,\alpha)\sum_{i=1}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i-1}}{(i-1)!}A^iX_0\right\}.$$

Therefore,

$$\begin{aligned}
D^{\alpha}_{\beta}X_{\alpha\beta}(t) &= (m'_t(t,\beta))^{-1}k(t,\alpha)m'_t(t,\beta)k^{-1}(t,\alpha)\sum_{i=1}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i-1}}{(i-1)!}A^iX_0\right\}\\
&= \sum_{i=1}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i-1}}{(i-1)!}A^iX_0\right\}\\
&= AX_0 + A\sum_{i=1}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i}}{(i)!}A^iX_0\right\}\\
&= AX_{\alpha\beta}(t),
\end{aligned}$$

i.e., $D^{\alpha}_{\beta}X_{\alpha\beta}(t) = AX_{\alpha\beta}(t)$. □

Remark 3.1. Since $(A+B)X_0 \subseteq AX_0 + BX_0$, where $A, B \in \mathbb{R}^{n\times n}$, $X_0 \in conv(\mathbb{R}^n)$, we have

$$\begin{aligned}
Y_{\alpha\beta}(t) = \left(\sum_{i=0}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i}}{i!}A^i\right\}\right)X_0 = e^{\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\,A}X_0\\
\subseteq X_0 + \sum_{i=1}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i}}{i!}A^iX_0\right\} = X_{\alpha\beta}(t).
\end{aligned}$$

It is easy to verify that $Y_{\alpha\beta}(t)$ is a solution to the following Cauchy problem:

$$D^{\alpha}_{\beta}y(t) = Ay(t), \quad y(0) \in X_0,$$

where $y \in \mathbb{R}^n$.

Note also that the equality $X_{\alpha\beta}(t) = Y_{\alpha\beta}(t)$ may hold, for example, if the matrix A has coinciding singular values and the set X_0 is a ball of radius r centered at $\mathbf{0}$.

Corollary 1. *If A is a positive-definite matrix, then*

$$X_{\alpha\beta}(t) = X_0 + U\sum_{i=1}^{\infty}\left\{\frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^{i}}{i!}\Sigma^iZ_0\right\},$$

where $t \in [0,T]$, $U\Sigma U^T$ is the singular-valued decomposition of the matrix A, $Z_0 = U^TX_0$.

Corollary 2. *If $k(t,\alpha) = t^{1-\alpha}$, $m(t,\beta) = t^{\beta}$ and $\alpha \in (0,1)$, $\beta = 1$, then we obtain the Cauchy problem for a differential equation with conformable fractional derivative of order α: $D^{\alpha}X(t) = AX(t)$, $X(0) = X_0$. Then the solution has the following form: $X(t) = X_0 + \sum\limits_{i=1}^{\infty}\{\frac{t^{i\alpha}}{\alpha^i i!}A^iX_0\}$.*

Corollary 3. *If $k(t,\alpha) = t^{1-\alpha}$, $m(t,\beta) = t^{\beta}$ and $\beta \in (0,1)$, $\alpha = 1$, then we obtain the Cauchy problem for a differential equation with fractal derivative of order β: $D_{\beta}X(t) = AX(t)$, $X(0) = X_0$. Then the solution has the following form: $X(t) = X_0 + \sum\limits_{i=1}^{\infty}\{\frac{t^{i\beta}}{i!}A^iX_0\}$.*

Corollary 4. *If $k(t,\alpha) = t^{1-\alpha}$, $m(t,\beta) = t^{\beta}$ and $\alpha = \beta = 1$, then we obtain the Cauchy problem for a differential equation with the Hukuhara derivative $D_HX(t) = AX(t)$, $X(0) = X_0$. Then the solution has the following form: $X(t) = X_0 + \sum\limits_{i=1}^{\infty}\{\frac{t^{i}}{i!}A^iX_0\}$.*

4 Impulsive linear set-valued differential equation with conformable fractional-fractal derivative

Consider the Cauchy problem for an impulsive linear set-valued differential equation of order (α,β)

$$D^{\alpha}_{\beta}X(t) = AX(t), \quad t \in [0,T] \setminus \{t_1,\dots,t_p\}, \tag{4.1}$$

$$X(0) = X_0, \quad X(t_i) = C_iX(t_i - 0), \quad t_i \in \{t_1,\dots,t_p\}, \tag{4.2}$$

where $\alpha,\beta \in (0,1]$, $X : [0,T] \to conv(\mathbb{R}^n)$ is an unknown set-valued mapping, $A, C_1,\dots,C_p \in \mathbb{R}^{n\times n}$ are nondegenerate matrices, $X_0 \in conv(\mathbb{R}^n)$, $0 < t_1 < t_2 < \cdots < t_p < T$ are fixed moments of time.

Definition 4.1. A set-valued mapping $X_{\alpha\beta} : [0,T] \to conv(\mathbb{R}^n)$ is called a solution of the Cauchy problem (4.1), (4.2) if it is piecewise-continuous and satisfies the differential equation (4.1)) for all $t \in [0,T] \setminus \{t_1,\dots,t_p\}$, $X_{\alpha\beta}(t_i) = C_iX_{\alpha\beta}(t_i - 0)$ for all $t \in \{t_1,\dots,t_p\}$ and $X(0) = X_0$.

Theorem 4.1. *The Cauchy problem (4.1), (4.2) has the following solution:*

$$X_{\alpha\beta}(t) = \begin{cases} \sum\limits_{i=0}^{\infty} \dfrac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^i}{i!} A^iX_0, & t \in [0,t_1), \\ \sum\limits_{i=0}^{\infty} \left\{ \dfrac{\left[\int\limits_{t_l}^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^i}{i!} A^i \right. & \\ \quad \times C_l \sum\limits_{i=0}^{\infty} \left\{ \dfrac{\left[\int\limits_{t_{l-1}}^{t_l} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^i}{i!} A^i \times \cdots \right. & \\ \quad \left. \left. \times C_1 \sum\limits_{i=0}^{\infty} \left\{ \dfrac{\left[\int\limits_0^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^i}{i!} A^iX_0 \right\} \cdots \right\} \right\}, & t \ge t_1, \end{cases}$$

where $t \in [0,T]$,

$$l = \begin{cases} 0, & t \in [0,t_1), \\ \max\limits_{i\in\{1,\dots,p\}} \{i : \ t_i \le t\}, & t \in [t_1,T]. \end{cases}$$

Proof. By Theorem 3.1, if $t \in [0,t_1)$, then

$$X_{\alpha\beta}(t) = \sum_{i=0}^{\infty} \left\{ \frac{\left[\int\limits_0^t m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^i}{i!} A^iX_0 \right\}.$$

By condition (4.2) and property 4) of Lemma 2.1, we have

$$X_{\alpha\beta}(t_1) = C_1 \sum_{i=0}^{\infty} \left\{ \frac{\left[\int\limits_0^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\right]^i}{i!} A^iX_0 \right\}.$$

For all $t \in [t_1, t_2)$, we have

$$X_{\alpha\beta}(t) = \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{t_1}^{t} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i \times C_1 \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{0}^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i X_0 \right\}\right\}$$

and

$$D^{\alpha}_{\beta}\left(\sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{t_1}^{t} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i \times C_1 \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{0}^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i X_0 \right\}\right\}\right)$$
$$= \sum_{i=1}^{\infty} \left\{ \frac{\big[\int_{t_1}^{t} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^{i-1}}{(i-1)!} A^i \times C_1 \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{0}^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i X_0 \right\}\right\}$$
$$= A\sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{t_1}^{t} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i \times C_1 \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{0}^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i X_0 \right\}\right\}.$$

Hence, for all $t \in [t_1, t_2)$, $D^{\alpha}_{\beta}(X_{\alpha\beta}(t)) = AX_{\alpha\beta}(t)$ and

$$X_{\alpha\beta}(t_2) = C_2\sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{t_1}^{t_2} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i \times C_1 \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{0}^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i X_0 \right\}\right\}.$$

And so on. □

Corollary 5. *If $AC_i = C_iA$ for all $i \in \{1,\dots,p\}$, then the Cauchy problem (4.1), (4.2) has the following solution:*

$$X_{\alpha\beta}(t) = \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\big[\int_{0}^{t} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i X_0, & t \in [0,t_1), \\ \displaystyle\sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{t_l}^{t} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i \right. & \\ \displaystyle\qquad \times \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{t_{l-1}}^{t_l} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i \times \cdots \right. & \\ \displaystyle\qquad \left. \left. \times \sum_{i=0}^{\infty} \left\{ \frac{\big[\int_{0}^{t_1} m'_s(s,\beta)k^{-1}(s,\alpha)\,ds\big]^i}{i!} A^i C_l \cdots C_1 X_0 \right\} \cdots \right\} \right\}, & t \geq t_1, \end{cases}$$

where $t \in [0, T]$,

$$l = \begin{cases} 0, & t \in [0,t_1), \\ \max\limits_{i\in\{1,\dots,p\}} \{i : \ t_i \leq t\}, & t \in [t_1, T], \end{cases}$$

Conclusion

It is well known that single-valued linear differential equations with conformable fractional and fractal derivatives are employed to model physical, biological, and engineering processes characterized by memory and self-similarity. Such equations have been extensively applied to anomalous diffusion, viscoelasticity, electrical circuits with memory, pharmacokinetics, financial time series, and control systems. The Cauchy problem addressed in this article extends these results to the cases where the initial state is not uniquely defined. Moreover, the obtained results can be readily generalized to situations in which the initial set is fuzzy.

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