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**DEEP NEURAL NETWORK APPROACH  
FOR PARABOLIC PDEs WITH NONSMOOTH  
INTERACTION FUNCTIONS**

**Abstract.** In this paper, we propose a theoretical framework for approximating mild solutions of parabolic partial differential equations (PDEs) with nonsmooth interaction functions using deep neural networks.

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**Key words and phrases.** Semilinear parabolic equation, mild solutions, Galerkin method, deep neural networks.

**რეზიუმე.** ნაშრომში განხილულია თეორიული საფუძველი დრმა ნეირონული ქსელების გამოყენებით არაგლუვი ურთიერთქმედების ფუნქციების მქონე პარაბოლური ნაწილობრივი დიფერენციალური განტოლებების (PDE) სუსტი ამონახსნების მიახლოებითი შეფასებისთვის.

## 1 Introduction

Deep learning algorithms are widely used tools for approximating solutions of various classes of PDEs [5]. In particular, the Deep Galerkin Method (DGM) employs a deep neural network instead of a linear combination of basis functions, and is trained to satisfy the differential equation, boundary conditions, and initial conditions by minimizing an error functional. Such an approach has been applied to nonlinear PDEs, including quasilinear parabolic equations, which admit classical solutions [11]. More recently, the DGM algorithm has been adapted to approximate weak solutions of parabolic inclusions, whose multivalued right-hand side can be represented by saturation functions [6].

In the present paper, we combine the ideas of [6, 11] and provide a theoretical foundation for approximating mild solutions of semilinear parabolic PDEs with continuous, but not necessarily smooth, nonlinear terms.

## 2 Statement of the problem

We consider the following semilinear parabolic problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(u), & (t, x) \in Q_T = (0, T) \times \Omega, \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x), \end{cases} \quad (2.1)$$

where  $u = u(t, x)$  is the unknown function,  $\Omega \subset \mathbb{R}^d$  is a bounded domain with a sufficiently smooth boundary  $\partial\Omega$ ,  $\Delta$  is the Laplace operator.

The nonlinear term  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, with  $f(0) = 0$ , and satisfies

$$f(s) s \leq C_1 s^2 + C_2, \quad s \in \mathbb{R}, \quad (2.2)$$

for some constants  $C_1, C_2 > 0$ .

In the phase space

$$X = C_0(\Omega) = \{v \in C(\bar{\Omega}) : v = 0 \text{ on } \partial\Omega\},$$

with the norm  $\|v\|_\infty = \sup_{x \in \Omega} |v(x)|$ , we seek an approximation of the *mild solution* of (2.1), defined as a function  $u \in C([0, T]; X)$  with  $u(0) = u_0$ , such that for all  $t \in [0, T]$

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s)) ds,$$

where  $T(t)$  is a  $C_0$ -semigroup of contractions generated by  $\Delta$  in  $X$ .

We prove that such a solution exists (see Theorem 3.1 below), but may be not unique. Our goal is to establish a theoretical result on the approximation power of Neural Networks (NN) for the solution given by Theorem 3.1.

We consider NNs with a single hidden layer and  $N$  hidden units from the class

$$W^N = \left\{ v(t, x) : \mathbb{R}^{1+d} \rightarrow \mathbb{R} \mid v(t, x) = \sum_{i=1}^N \alpha_i \sigma \left( \theta_{0,i} t + \sum_{j=1}^d \theta_{j,i} x_j + c_i \right) \right\}, \quad (2.3)$$

where  $\sigma$  is a nonlinear smooth activation function, and  $\theta = \{\alpha_i, \theta_{j,i}, c_i\} \in \mathbb{R}^K$  are the NN's parameters minimizing the objective functional (the  $L^2$ -error):

$$J(v) = \left\| \frac{\partial v}{\partial t} - \Delta v - f(v) \right\|_{L^2(Q_T)}^2 + \|v\|_{L^2((0,T) \times \partial\Omega)}^2 + \|v(0, \cdot) - u_0\|_{L^2(\Omega)}^2. \quad (2.4)$$

Note that this functional is not defined on mild solutions of (2.1), but we can directly compute  $J(v)$  for any  $v \in W^N$ . The smaller the value of  $J(v)$ , the better the corresponding function  $v$  approximates a solution of (2.1).

To justify the above procedure, we rely on the following key results:

- the density of  $W = \bigcup_{N \geq 1} W^N$  in  $C^2(\overline{Q}_T)$  (see Theorem 3.2 below),
- the uniform approximation of  $f$  by  $C^2$ -functions while preserving property (2.2) (see Theorem 3.3 below).

Our main results are as follows:

1. There exists  $v^N \in W^N$  such that

$$J(v^N) \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (2.5)$$

2. If a sequence  $\{v^N\} \subset W$  satisfies (2.5), then, up to a subsequence,

$$v^N \rightharpoonup u \text{ in } L^2(Q_T),$$

where  $u$  is a mild solution of (2.1).

### 3 Main results

First, we establish the mild solvability of (2.1) in  $X$ . Under different assumptions, this result was proved in [2, 9].

**Theorem 3.1.** *Assume that condition (2.2) holds. Then, for every  $r > 0$  and every  $u_0 \in X$  with  $\|u_0\|_\infty \leq r$ , there exists a mild solution of (2.1), and for all  $t \in [0, T]$ ,*

$$\|u(t)\|_\infty \leq R = R(r). \quad (3.1)$$

*Proof.* Since  $T(t)$  is a compact semigroup and  $F : X \rightarrow X$ ,  $F(u)(x) = f(u(x))$ , is continuous, it follows from [8, Chapter 6, Theorem 2.1] that for every  $u_0 \in X$  there exists a (local) mild solution on some interval  $[0, t_1]$ , with  $t_1 = t_1(u_0) > 0$ .

We now prove estimate (3.1) that guarantees the existence on  $[0, T]$ . It is known [3] that  $T(t)$  satisfies the following inequalities:

$$\begin{aligned} \exists C > 0, \quad \alpha \in (0, 1), \quad \delta \in \left(\frac{1}{2}, 1\right) \text{ such that } \forall u_0 \in L^2(\Omega), \quad \|T(t)u_0\|_{H^2} \leq \frac{C}{t} \|u_0\|_{L^2}, \\ \forall u_0 \in X, \quad \|T(t)u_0\|_{C^{1+\alpha}} \leq \frac{C}{t^\delta} \|u_0\|_\infty, \quad \exists M > 1, \quad \lambda > 0 \text{ such that } \|T(t)\|_{L(X, X)} \leq Me^{-\lambda t}. \end{aligned} \quad (3.2)$$

Now, let  $v = v(t, x)$  be a mild solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v - \beta v + f(v e^{\beta t}) e^{-\beta t}, \\ v|_{\partial\Omega} = 0, \quad v|_{t=0} = u_0(x), \end{cases} \quad (3.3)$$

where  $\beta > 0$ . Such a solution also exists locally on  $[0, \tau]$  due to [8, Chapter 6, Theorem 2.1]. Then  $u(t) = v(t) e^{\beta t}$  is a mild solution of (2.1). Indeed,

$$v(t) = T(t)u_0 + \int_0^t T(t-s)(f(v(s))e^{-\beta s} - \beta v(s)) ds.$$

So,

$$u(t) = T(t)e^{\beta t}u_0 + \int_0^t T(t-s)e^{\beta(t-s)}(f(u(s)) - \beta u(s)) ds,$$

and the required result follows from the fact that  $T(t)e^{\beta t}$  is the  $C_0$ -semigroup generated by  $\Delta + \beta I$  in  $X$ . Therefore, it is sufficient to show the mild solvability of (3.3) on  $[0, T]$ .

Let us choose  $\beta \geq 2C_1$ . Then, for all  $t \in [0, T]$  and all  $s \in \mathbb{R}$ ,

$$(-\beta s + f(se^{\beta t})e^{-\beta t})s \leq -\beta s^2 + C_1 s^2 + C_2 \leq -C_1 s^2 + C_2 \leq 0, \quad \text{if } |s| \geq M := \max \left\{ \left( \frac{C_2}{C_1} \right)^{\frac{1}{2}}, r \right\}. \quad (3.4)$$

Using (3.2), we have that for a mild solution  $v$  of (3.3), for all  $t \in (0, \tau]$ ,

$$\begin{aligned} \|v(t)\|_{C^{1+\alpha}} &\leq Ct^{-\delta} \|u_0\|_{\infty} + \int_0^t Cs^{-\delta} \|f_1(t-s, v(t-s))\|_{\infty} ds \\ &\leq Ct^{-\delta} \|u_0\|_{\infty} + \frac{Ct^{1-\delta}}{1-\delta} \max_{s \in [0, \tau]} \|f_1(t, v(t))\|_{\infty}, \end{aligned} \quad (3.5)$$

where  $f_1(t, v) = -\beta v + f(v e^{\beta t}) e^{-\beta t}$ .

Let us denote  $h(t) = f_1(t, v(t))$ . Then, due to [1],  $v$  is a weak solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v + h(t), \\ v|_{t=0} = u_0 \end{cases}$$

on  $[0, \tau]$ . Due to (3.5),  $\forall t_0 \in (0, \tau)$ ,  $v(t_0) \in H_0^1(\Omega)$ , which means that

$$v \in C([t_0, \tau]; H_0^1(\Omega)) \cap L^2(t_0, T; H^2(\Omega) \cap H_0^1(\Omega)), \quad v_t \in L^2(t_0, \tau; L^2(\Omega)).$$

These regularity results allow us to apply the truncation function approach: multiply (3.2) on  $(t_0, \tau)$  by

$$(v - M)_+ = \begin{cases} v - M, & v \geq M, \\ 0, & v \leq M \end{cases}$$

in  $L^2(\Omega)$ , where  $M$  is taken from (3.4). Then from (3.4) we have

$$\frac{1}{2} \frac{d}{dt} \|(v - M)_+\|^2 + \lambda_1 \|(v - M)_+\|^2 \leq \int_{\Omega} f_1(t, v(t, x))(v(t, x) - M)_+ dx \leq 0, \quad (3.6)$$

where  $\lambda_1 > 0$  is taken from the Poincaré inequality.

Therefore, for all  $t \in [t_0, \tau]$ ,

$$\int_{\Omega} (v(t, x) - M)_+^2 dx \leq e^{-2\lambda_1 t} \int_{\Omega} (v(t_0, x) - M)_+^2 dx.$$

Due to the continuity of  $v : [0, \tau] \mapsto X$ , we can pass to the limit as  $t_0 \rightarrow 0$  and from (3.6) obtain

$$\forall t \in [0, \tau], \quad \forall x \in \Omega, \quad v(t, x) \leq M.$$

Repeating the same arguments for  $(v - M)_-$ , we get

$$\forall t \in [0, T], \quad v(t, x) \geq -M.$$

So, for  $u(t, x) = v(t, x)e^{\beta t}$ , we have the existence on  $[0, T]$  with the inequality

$$\forall t \in [0, T], \quad \|u(t)\|_{\infty} \leq R = Me^{\beta T}. \quad \square$$

Second, we need some auxiliary approximation results.

We introduce the class

$$W = \bigcup_{N \geq 1} W^N, \quad (3.7)$$

where  $W^N$  was defined in (2.3).

**Theorem 3.2** ([4]). *For every  $\varepsilon > 0$  and  $u \in C^{1,2}([0, T] \times \overline{\Omega})$ , there exists  $v \in W$  such that*

$$\|u - v\|_{C^{1,2}(\overline{Q_T})} < \varepsilon. \quad (3.8)$$

**Theorem 3.3.** *Assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $f(0) = 0$ , and with some  $C_1, C_2 \geq 0$ ,*

$$f(s) \cdot s \leq C_1 s^2 + C_2, \quad \forall s \in \mathbb{R}.$$

*Then there exists a sequence  $\{f_n : \mathbb{R} \rightarrow \mathbb{R}\}$  of  $C^\infty$ -functions such that  $f_n(0) = 0$ , and*

$$f_n(s) \cdot s \leq \tilde{C}_1 s^2 + \tilde{C}_2, \quad \forall s \in \mathbb{R},$$

*where positive constants  $\tilde{C}_1, \tilde{C}_2$  do not depend on  $n$ , and*

$$\sup_{s \in \mathbb{R}} |f_n(s) - f(s)| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.9)$$

*Proof.* First, we prove that for  $f \in C([0, 1])$  there exists  $\{f_n\} \subset C^\infty([0, 1])$  such that

$$\max_{s \in [0, 1]} |f_n(s) - f(s)| \rightarrow 0, \quad n \rightarrow \infty, \quad (3.10)$$

and

$$f_n(0) = f(0), \quad f_n(1) = f(1), \quad f_n^{(j)}(0) = f_n^{(j)}(1) = 0, \quad j \geq 1.$$

Indeed, let  $\varphi \in C^\infty(\mathbb{R})$  be a strictly increasing function on  $[0, 1]$  such that

$$\varphi(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x \geq 1. \end{cases}$$

Then, using the Bernstein polynomials, it is easy to see that

$$f_n(x) := \sum_{k=0}^n f\left(\varphi^{-1}\left(\frac{k}{n}\right)\right) \binom{n}{k} \varphi^k(x) (1 - \varphi(x))^{n-k}$$

satisfies (3.10). Now, let us construct a sequence  $\{f_n\} \subset C^\infty(\mathbb{R})$  such that (3.9) holds. We take arbitrary  $\varepsilon > 0$  and for every  $k \in \mathbb{Z}$  we consider

$$g_{k,\varepsilon}(t) = f(k+t), \quad t \in [0, 1].$$

From the first part of the proof, we deduce the existence of  $g_{k,\varepsilon} \in C^\infty([0, 1])$  such that

$$g_{k,\varepsilon}(0) = f(k), \quad g_{k,\varepsilon}(1) = f(k+1), \quad g_{k,\varepsilon}^{(j)}(0) = g_{k,\varepsilon}^{(j)}(1) = 0, \quad j \geq 1,$$

and

$$\max_{t \in [0, 1]} |g_k(t) - g_{k,\varepsilon}(t)| < \varepsilon.$$

We define

$$f_\varepsilon(x) := g_{k,\varepsilon}(x - k), \quad x \in [k, k+1), \quad k \in \mathbb{Z}.$$

Then  $f_\varepsilon \in C^\infty(\mathbb{R})$ ,  $f_\varepsilon(k) = f(k)$  for all  $k \in \mathbb{Z}$ , in particular  $f_\varepsilon(0) = 0$ , and  $\forall x \in \mathbb{R}$ , we have  $x \in [k, k+1)$ , so,

$$|f(x) - f_\varepsilon(x)| = |g_k(x - k) - g_{k,\varepsilon}(x - k)| < \varepsilon.$$

Finally, the inequality

$$\forall x \in \mathbb{R}, \quad \forall n \geq 1, \quad f(x) - 1 \leq f_n(x) \leq f(x) + 1,$$

which holds due to (3.8), guarantees the property

$$\forall s \in \mathbb{R}, \quad f_n(s) \cdot s \leq \tilde{C}_1 s^2 + \tilde{C}_2,$$

with constants  $\tilde{C}_1, \tilde{C}_2$  that do not depend on  $n$ . □

**Theorem 3.4.** *Assume that  $u_0 \in X \cap H_0^1(\Omega)$ . Then, for the functional  $J$  defined in (2.4) and the set  $W$  defined in (3.7), we have:  
there exists  $\{v^N\} \subset W$  such that*

$$J(v^N) \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (3.11)$$

*Proof.* We consider the following auxiliary problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f_n(u), & (t, x) \in Q_T, \\ u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0^n(x), \end{cases} \quad (3.12)$$

where  $C^\infty$ -functions  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  are taken from Theorem 3.3,  $u_0^n \in C_0^\infty(\Omega)$ ,  $u_0^n \rightarrow u_0$  in  $X \cap H_0^1$  and  $\|u_0^n\|_\infty \leq r$ .

Since  $f_n$  satisfies (2.2), it follows from [7, Theorem 6.1] that for every  $n \geq 1$ , problem (3.12) has a unique classical solution  $u^n \in C^{1,2}(Q_T)$ , which is, of course, also a mild solution.

Therefore, the functional

$$J_n(v) = \left\| \frac{\partial v}{\partial t} - \Delta v - f_n(v) \right\|_{L^2(Q_T)}^2 + \|v\|_{L^2((0,T) \times \partial\Omega)}^2 + \|v(0, \cdot) - u_0\|_{L^2(\Omega)}^2$$

takes its minimal value 0 for  $v = u^n$ .

Moreover, due to Theorem 3.1 and Theorem 3.3, we have

$$\max_{t \in [0, T]} \|u^n(t)\|_\infty \leq \tilde{R}, \quad (3.13)$$

where  $\tilde{R} > 0$  does not depend on  $n$ . In what follows, we denote by  $R$  the maximum of the constants given in (3.1) and (3.13).

For every  $v \in C^{1,2}(\bar{Q}_T)$ ,  $\max_{t \in [0, T]} \|v(t)\|_\infty \leq R$ , we have

$$\begin{aligned} |J_n(v) - J(v)| &= \left| \left\| \frac{\partial v}{\partial t} - \Delta v - f_n(v) \right\|_{L^2(Q_T)}^2 - \left\| \frac{\partial v}{\partial t} - \Delta v - f(v) \right\|_{L^2(Q_T)}^2 \right| \\ &\quad + \left| \|v(0, \cdot) - u_0^n\|_{L^2(\Omega)}^2 - \|v(0, \cdot) - u_0\|_{L^2(\Omega)}^2 \right| \\ &\leq M_1(R) \|f_n(v) - f(v)\|_{L^2(Q_T)} + M_2(v) \|f_n(v) - f(v)\|_{L^2(\Omega)} \\ &\quad + M_3(R) \|u_0^n - u_0\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.14)$$

where  $M_1(R)$ ,  $M_3(R)$  depend only on  $R$ , and  $M_2(v)$  depends on  $\|v_t\|_{L^2(Q_T)} + \|\Delta v\|_{L^2(Q_T)}$ .

Now, let  $u^n$  be the solution of (3.12). According to Theorem 3.2, there exist  $N = N(n)$  and  $v^N \in W^N$  such that

$$\|u^n - v^{N(n)}\|_{C^{1,2}(\bar{Q}_T)} < \frac{1}{nL_nC_n}, \quad (3.15)$$

where

$$L_n = \max_{|s| \leq 2R} |f'_n(s)|, \quad C_n = \|u^n\|_{C^{1,2}(Q_T)}.$$

In particular, we can assume that

$$\sup_{t \in [0, T]} \|v^N(t)\|_\infty \leq 2R.$$

Then

$$|J_n(u^n) - J_n(v^{N(n)})| = |J_n(v^{N(n)})| \leq \frac{C}{n}, \quad (3.16)$$

where  $C$  does not depend on  $n$ .

So,

$$|J(v^{N(n)})| \leq |J_n(v^{N(n)}) - J(v^{N(n)})| + |J_n(v^{N(n)})|,$$

and, due to (3.16), we need to prove that the second summand tends to zero as  $n \rightarrow \infty$ .

From (3.14), (3.15) we deduce that

$$\begin{aligned} |J_n(v^{N(n)}) - J(v^{N(n)})| &\leq M_1(2R) \|f_n(v^{N(n)}) - f(v^{N(n)})\|_{L^2(Q_T)}^2 + M_3(2R) \|u_0^n - u_0\|_{L^2(\Omega)}^2 \\ &\quad + C \left(1 + \|u_t^n\|_{L^2(Q_T)} + \|\Delta u^n\|_{L^2(Q_T)}\right) \|f_n(v^{N(n)}) - f(v^{N(n)})\|_{L^2(Q_T)}. \end{aligned} \quad (3.17)$$

According to (3.9),

$$\|f_n(v^{N(n)}) - f(v^{N(n)})\|_{L^2(Q_T)} \rightarrow 0, \quad n \rightarrow \infty.$$

The inclusion  $u_0 \in H_0^1(\Omega)$  implies the boundedness of  $\{u_0^n\}$  in  $H_0^1(\Omega)$ . This fact, together with the boundedness of  $\{f_n(u^n(t, x))\}$  in  $L^2(Q_T)$ , implies (see [12, Chapter II, Theorem 3.3]) that

$$\{u^n\} \text{ is bounded in } L^2(0, T; H^2(\Omega)), \quad \{u_t^n\} \text{ is bounded in } L^2(Q_T). \quad (3.18)$$

Therefore, (3.16)–(3.18) implies (3.11).  $\square$

Note that the convergence  $J(v^N) \rightarrow 0$  does not directly imply that  $v^N$  converges to a solution of (2.1) in any specific sense.

*Remark.* From the proof of Theorem 3.4 (see (3.15), (3.18)), we can see that the sequence  $\{v^N\} \subset W$  in (3.11) satisfies the following property:

$$\{v^N\} \text{ is bounded in the norm of } L^2(0, T; H^2) \text{ and in } L^\infty(Q_T), \quad (3.19)$$

$$\{v_t^N\} \text{ is bounded in the norm of } L^2(Q_T). \quad (3.20)$$

**Theorem 3.5.** *Assume that  $u_0 \in X \cap H_0^1(\Omega)$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies (2.2) and is locally Lipschitz. Assume that  $\{v^N\} \subset W$  satisfies (3.19), (3.20), and*

$$J(v^N) \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (3.21)$$

Then

$$v^N \rightarrow u \text{ in } L^2(Q_T),$$

where  $u$  is a mild solution of (2.1).

*Proof.* From (3.21) we can deduce that every  $v^N$  is a classical solution of the initial boundary value problem

$$\begin{cases} \frac{\partial v^N}{\partial t} = \Delta v^N + f(v^N) + e^N(t, x), & (t, x) \in Q_T, \\ v^N|_{\partial\Omega} = g^N(t, x), & (t, x) \in (0, T) \times \partial\Omega, \\ v^N|_{t=0} = v_0^N(x), \end{cases} \quad (3.22)$$

where  $e^N, g^N, v_0^N$  are  $C^\infty$ -functions such that

$$\|e^N\|_{L^2(Q_T)}^2 + \|g^N\|_{L^2((0, T) \times \partial\Omega)}^2 + \|v_0^N - u_0\|_{L^2(\Omega)}^2 \rightarrow 0. \quad (3.23)$$

Let  $u^N$  be a solution of (3.22) with zero boundary conditions, i.e.,  $u^N|_{\partial\Omega} = 0$ . Such a solution exists in the mild sense by Theorem 3.1. Moreover, it is unique due to the Lipschitz continuity of  $f$ . According to [1],  $u^N$  is a weak solution of (3.22), and from (2.2) we deduce the estimate

$$\frac{1}{2} \frac{d}{dt} \|u^N(t)\|_{L^2}^2 + \|\nabla u^N\|_{L^2}^2 \leq C_1 \|u^N(t)\|_{L^2}^2 + C_2 |\Omega| + (e^N(t), u^N(t))_{L^2}.$$

After applying Gronwall's Lemma, we have

$$\begin{aligned} \{u^N\} &\text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \\ \{u_t^N\} &\text{ is bounded in } L^2(0, T; H^{-1}(\Omega)). \end{aligned} \quad (3.24)$$

Relation (3.24) implies that, up to a subsequence,  $u^N$  and  $u_t^N$  converge weakly to some functions  $u$  and  $u_t$  in the corresponding spaces. Moreover, the Compactness Theorem [10, Theorem 8.1] guarantees that, up to a subsequence,

$$u^N \rightarrow u \text{ in } L^2(Q_T), \text{ and a.e. on } Q_T.$$

In particular,

$$f(u^N(t, x)) \rightarrow f(u(t, x)) \text{ a.e. on } Q_T. \quad (3.25)$$

From (3.23) and Green's formula, we can assume that  $\{v_0^N\}$  is bounded in  $H^1(\Omega)$ . Thus,  $u^N$  satisfies (3.18), which means that  $\{f(u^N)\}$  is bounded in  $L^2(Q_T)$ . Combining this result with (3.25), we deduce from [10, Lemma 8.3] that  $f(u^N) \rightarrow f(u)$  weakly in  $L^2(Q_T)$ .

Thus, we can pass to the limit in the equality

$$\int_{Q_T} \frac{\partial u^N}{\partial t} \zeta + \int_{Q_T} \nabla_x u^N \cdot \nabla_x \zeta(t, x) = \int_{Q_T} f(u^N(t, x)) \zeta(t, x) + \int_{Q_T} e^N(t, x) \zeta(t, x), \quad \zeta \in L^2(0, T; H_0^1(\Omega)),$$

and obtain that  $u$  is a weak solution of (2.1).

Since  $u_0 \in H_0^1(\Omega) \cap X$  and  $f(u) \in L^2(Q_T)$ , we deduce from [1] that  $u$  is a mild solution of (2.1).

Finally, let us estimate  $v^N - u^N$ . Using (3.23) and Green's formula, we have

$$\frac{d}{dt} \|v^N - u^N\|_{L^2}^2 + \|\nabla(v^N - u^N)\|_{L^2}^2 \leq \int_{\partial\Omega} g^N \cdot \nabla(v^N - u^N) + \int_{\partial\Omega} (f(u^N) - f(v^N))(v^N - u^N). \quad (3.26)$$

Taking into account that  $f$  is locally Lipschitz and  $\{v^N - u^N\}$  is bounded in  $L^2(0, T; H^2)$ , we deduce from (3.26) that

$$v^N - u^N \rightarrow 0 \text{ in } L^2(Q_T). \quad \square$$

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