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# AVERAGING METHOD FOR MINIMAX ESTIMATION OF FUNCTIONALS OF SOLUTIONS TO PARABOLIC PROBLEMS

*Dedicated to the blessed memory of Professor M. O. Perestyuk*

**Abstract.** We study the problem of minimax estimation of functionals depending on the solutions of parabolic boundary value problems, where the observation model is based on time-averaging. Using the averaging method, we construct approximate minimax estimates and analyze the corresponding estimation errors. The obtained results show that the averaging approach provides effective tools for reducing the complexity of minimax estimation in parabolic systems with rapidly oscillating coefficients.

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**Key words and phrases.** Averaging method, observation, minimax estimation, parabolic problem.

**რეზიუმე.** შესწავლილია პარაბოლური სასაზღვრო ამოცანების ამონახსნებზე დამოკიდებული ფუნქციონალების მინიმაქსური შეფასების პრობლემა, სადაც დაკვირვების მოდელი ეფუძნება დროის მიმართ გასაშუალოებას. გასაშუალოების მეთოდის გამოყენებით აგებულია მიახლოებითი მინიმაქსური შეფასებები და გაანალიზებულია შეფასებების შესაბამისი ცდომილებები. მიღებული შედეგები აჩვენებს, რომ გასაშუალების მიდგომა იძლევა ეფექტურ ინსტრუმენტებს სწრაფად რხევად კოეფიციენტებიან პარაბოლურ სისტემებში მინიმაქსური შეფასების სირთულის შესამცირებლად.

# 1 Introduction

Minimax estimation of functionals arising in partial differential equations (PDEs) has a long history. O. Nakonechnyi first formulated the general minimax estimation problem for operator equations in Hilbert spaces [11]. In the parabolic PDE setting, O. Kapustian and O. Nakonechnyi examined point-wise observation problems for boundary-value problems [5], and later synthesized optimal bounded controls for parabolic systems with rapidly oscillating coefficients [3]. Subsequent works developed guaranteed mean-square estimates using homogenization: for example, O. Nakonechnyi et al. proved the existence of a guaranteed linear estimate and used averaging theory to construct approximate minimax solutions for parabolic equations with fast oscillations [10]. Hyperbolic systems have been treated similarly: O. Kapustian et al. in [4] extended minimax estimation techniques to linear hyperbolic PDEs under uncertainty. In addition to the works mentioned above, several further contributions should be noted. T. Basar and P. Bernhard [1] developed the general framework of $H^\infty$-optimal control and related minimax design problems within a dynamic game approach, establishing the modern mathematical foundations of minimax control. M. Luz and M. Moklyachuk [9] studied minimax-robust filtering for stochastic sequences, extending the minimax estimation paradigm to cointegrated and nonstationary models. A. M. Samoilenko and N. A. Perestyuk [13] presented a comprehensive monograph on impulsive differential equations, where averaging and robustness methods play a crucial role in stability and control analysis. Finally, A. Bensoussan [2] provided a systematic exposition of perturbation and averaging methods in optimal control, which strongly influenced subsequent research on minimax estimation and control of distributed parameter systems.

Apart from estimation, related contributions have been addressed control problems for parabolic systems. For instance, in [6] the authors developed an approximate bounded synthesis method for distributed control of parabolic models. Averaging methods have also been applied in this context: O. V. Kapustyan et al. recently used a time-averaging method in optimal control for parabolic differential inclusions with rapidly oscillating coefficients [7]. In general, the averaging, either in time or in parameters, can simplify complex control problems [12].

In the present paper, we study the problem of minimax estimation of functionals depending on the solution of a parabolic boundary value problem, where the observation model involves averaging over time. This distinguishes our approach from previous works based on spatial homogenization.

# 2 Setting of the problem

The paper studies the problem of approximate minimax estimation of solutions of parabolic problems. Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ $(n \geq 1)$. In the cylinder $Q_T = \Omega \times (0,T)$, we consider the problem

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + A\varphi = f_1(x,t), \\ \varphi\big|_{\partial\Omega} = 0, \\ \varphi\big|_{t=0} = f_0(x), \end{cases} \tag{2.1}$$

where $A = -\operatorname{div}(a(x)\nabla)$, $a \in L^\infty(\Omega)$, is a symmetric matrix satisfying the uniform ellipticity conditions.

According to [8], for the fixed $f_1 \in L^2(Q_T)$, $f_0 \in L^2(\Omega)$, there exists a unique solution in the space $W(0,T) = \{y \in L^2(0,T;H_0^1(\Omega)) : \ y_t \in L^2(0,T;H^{-1}(\Omega))\}$.

The observed function is

$$y^\varepsilon(t) = \int\limits_\Omega C\Big(x, \frac{t}{\varepsilon}\Big)\varphi(x,t)\,dx + f_2(t), \tag{2.2}$$

where $C(x,\tau) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function, $C \in L^\infty(\Omega \times \mathbb{R})$, and there exists the limit

$$\lim_{T\to\infty} \frac{1}{T}\int\limits_0^T |C(x,s) - \overline{C}(x)|\,ds = 0 \ \text{ in } L^2(\Omega).$$

The functions $f_0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$, $f_2 \in L^2(0,T)$ are unknown and belong to a convex closed set $G$ of the space $L^2(\Omega) \times L^2(Q_T) \times L^2(0,T)$, namely,

$$\{f_0, f_1, f_2\} \in G = \Big\{ \|f_0\|^2_{L^2(\Omega)} + \|f_1\|^2_{L^2(Q_T)} + \|f_2\|^2_{L^2(0,T)} \le 1 \Big\}. \tag{2.3}$$

The main problem is to estimate the functional of the solutions of problem (2.1)

$$l(\varphi) = \int_0^T \int_\Omega a(x,t)\varphi(x,t)\, dx\, dt, \tag{2.4}$$

where $a \in L^2(Q_T)$, in the class of linear functionals of the observations

$$\widehat{\widehat{l}}(\varphi) = \int_0^T y^\varepsilon(t)u(t)\, dt, \tag{2.5}$$

where $u \in L^2(0,T)$ is the solution of the problem

$$J_\varepsilon(u) = \sup_{\{f_0,f_1,f_2\}\in G} \big(l(\varphi) - \widehat{\widehat{l}}(\varphi)\big)^2 \longrightarrow \inf. \tag{2.6}$$

We consider the quantity

$$\sigma_\varepsilon = \inf_u J_\varepsilon(u), \tag{2.7}$$

which is called the error of the minimax estimation.

# 3 Analysis of the averaged problem

For a fixed $u \in L^2(0,T)$, let us analyze the following averaged problem:

$$\begin{cases} \big(l(\varphi) - \widehat{\widehat{l}}_0(\varphi)\big)^2 \to \sup, \\ \{f_0, f_1, f_2\} \in G, \end{cases}$$

where $\varphi$ is the solution of problem (2.1), $\widehat{\widehat{l}}_0(\varphi) = \int_0^T u(t)y^0(t)\, dt$, with $y^0(t) = \int_\Omega \overline{C}(x)\varphi(x,t)\, dx + f_2(t)$.

Let $g(\varphi) := l(\varphi) - \widehat{\widehat{l}}_0(\varphi)$, or equivalently,

$$g(\varphi) = \int_0^T \int_\Omega a(x,t)\varphi(x,t)\, dx\, dt - \int_0^T u(t)y^0(t)\, dt \\ - \int_0^T \int_\Omega z(x,t)\big(\dot\varphi(x,t) - \operatorname{div}(b(x)\nabla\varphi(x,t)) - f_1(x,t)\big)\, dx\, dt =: I^1 + I^2 + I^3, \tag{3.1}$$

where $z$ is the solution of the following adjoint system:

$$\begin{cases} \dot z(x,t) - Az = -a(x,t) + \overline{C}(x)u, \\ z\big|_{\partial\Omega} = 0, \\ z\big|_{t=T} = 0. \end{cases} \tag{3.2}$$

*Remark* 1. Problem (3.2) indeed has a solution, since by substituting $s = T - t$ it reduces to problem (2.1) with the right-hand side of opposite sign and zero initial conditions.

Let us analyze $I^3$ in more detail:

$$I^3 = -\int_0^T \int_\Omega z(x,t)\dot{\varphi}(x,t)\,dx\,dt + \int_0^T \int_\Omega z(x,t)\,\mathrm{div}(b(x)\nabla\varphi(x,t))\,dx\,dt + \int_0^T \int_\Omega z(x,t) f_1(x,t)\,dx\,dt =: I_1 + I_2 + I_3. \tag{3.3}$$

By Green's formula, together with the initial and boundary conditions, we obtain

$$\begin{aligned} I_1 &= -\int_0^T \int_\Omega z(x,t)\dot{\varphi}(x,t)\,dx\,dt \\ &= -\int_\Omega z(x,t)\varphi(x,t)\big|_0^T\,dx + \int_0^T \int_\Omega \dot{z}(x,t)\varphi(x,t)\,dx\,dt \\ &= \int_\Omega z(x,0) f_0(x)\,dx + \int_0^T \int_\Omega \dot{z}(x,t)\varphi(x,t)\,dx\,dt, \end{aligned} \tag{3.4}$$

$$\begin{aligned} I_2 &= \int_0^T \int_\Omega z(x,t)\,\mathrm{div}(b(x)\nabla\varphi(x,t))\,dx\,dt \\ &= \int_0^T \left( \int_{\partial\Omega} z(x,t)b(x)\,\frac{\partial\varphi}{\partial n_A}\,d\xi\,dt - \int_\Omega b(x)\nabla\varphi(x,t)\nabla z(x,t)\,dx \right) \\ &= \int_0^T \int_\Omega b(x)\nabla\varphi(x,t)\nabla z(x,t)\,dx\,dt = \int_0^T \int_\Omega \varphi(x,t)\,\mathrm{div}(b(x)z(x,t))\,dx\,dt, \end{aligned} \tag{3.5}$$

where $\frac{\partial\varphi}{\partial n_A}$ denotes the derivative with respect to the outer normal.

From (3.3)–(3.5), we obtain

$$I^3 = \int_\Omega z(x,0) f_0(x)\,dx + \int_0^T \int_\Omega \dot{z}(x,t)\varphi(x,t)\,dx\,dt + \int_0^T \int_\Omega \varphi(x,t)\,\mathrm{div}(b(x)z(x,t))\,dx\,dt + \int_0^T \int_\Omega z(x,t) f_1(x,t)\,dx\,dt. \tag{3.6}$$

Taking into account (3.1), (3.6) and the fact that $z$ is the solution of (3.2), we get

$$\begin{aligned} g(\varphi) = \int_0^T \int_\Omega a(x,t)\varphi(x,t)\,dx\,dt - \int_0^T \int_\Omega u(t)\overline{C}(x)\varphi(x,t)\,dx\,dt \\ - \int_0^T u(t) f_2(t)\,dt + \int_\Omega z(x,0) f_0(x)\,dx + \int_0^T \int_\Omega \dot{z}(x,t)\varphi(x,t)\,dx\,dt \\ + \int_0^T \int_\Omega \varphi(x,t)\,\mathrm{div}(b(x)z(x,t))\,dx\,dt + \int_0^T \int_\Omega z(x,t) f_1(x,t)\,dx\,dt \end{aligned}$$

$$= \int_0^T \int_\Omega \varphi(x,t)\big(a(x,t) - \overline{C}(x)u(t) + \dot{z}(x,t) + \operatorname{div}(b(x)\nabla z(x,t))\big)\,dx\,dt$$

$$- \int_0^T u(t)f_2(t)\,dt + \int_\Omega z(x,0)f_0(x)\,dx + \int_0^T \int_\Omega z(x,t)f_1(x,t)\,dx\,dt$$

$$= - \int_0^T u(t)f_2(t)\,dt + \int_\Omega z(x,0)f_0(x)\,dx + \int_0^T \int_\Omega z(x,t)f_1(x,t)\,dx\,dt. \tag{3.7}$$

From the necessary conditions for the existence of an extremum of the functional

$$F = \big(l(\varphi) - \widehat{\widehat{l}}(\varphi)\big)^2 = g(\varphi)^2 = \Big( - \int_0^T u(t)f_2(t)\,dt + \int_\Omega z(x,0)f_0(x)\,dx + \int_0^T \int_\Omega z(x,t)f_1(x,t)\,dx\,dt\Big)^2,$$

we obtain

$$f_0 = \lambda z(x,0), \quad f_1 = \lambda z(x,t), \quad f_2 = -\lambda u, \tag{3.8}$$

where

$$\lambda^2 = \Big(\|u\|^2_{L^2(0,T)} + \|z(x,0)\|^2_{L^2(\Omega)} + \|z(x,t)\|^2_{L^2(Q_T)}\Big)^{-1}. \tag{3.9}$$

Therefore,

$$F(f_0,f_1,f_2) = \Big( - \int_0^T (-\lambda)u^2(t)\,dt + \int_\Omega \lambda z^2(x,0)\,dx + \int_0^T \int_\Omega \lambda z^2(x,t)\,dx\,dt\Big)^2$$

$$= \lambda^2\Big( \int_0^T u^2(t)\,dt + \int_\Omega z^2(x,0)\,dx + \int_0^T \int_\Omega z^2(x,t)\,dx\,dt\Big)^2$$

$$= \|u\|^2_{L^2(0,T)} + \|z(x,0)\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(Q_T)}.$$

Hence,

$$J^0(u) = \sup_{\{f_0,f_1,f_2\}\in G} \big(l(\varphi) - \widehat{\widehat{l}}(\varphi)\big)^2 = \|u\|^2_{L^2(0,T)} + \|z(x,0)\|^2_{L^2(\Omega)} + \|z\|^2_{L^2(Q_T)}. \tag{3.10}$$

Now, let us consider the following problem:

$$\sigma_0 = \inf_u J^0(u).$$

Introduce the function $p$ as the solution of the following problem:

$$\begin{cases} \dfrac{\partial p}{\partial t} + Ap = z, \\ p\big|_{\partial\Omega} = 0, \\ p(x,0) = z(x,0). \end{cases} \tag{3.11}$$

Multiplying the state equation (3.2) by $p$ and integrating over $Q_T$, we obtain

$$\int_0^T \int_\Omega p\Big(\frac{\partial z}{\partial t} + \operatorname{div}(b(x)z)\Big)\,dx\,dt = \int_0^T \int_\Omega p\big( - a(x,t) + \overline{C}(x)u(t)\big)\,dx\,dt.$$

Using integration by parts and Green's formula, and taking into account the boundary and initial conditions, from (3.2) and (3.11), we get

$$-\int_\Omega z^2(x,0)\,dx - \int_0^T\int_\Omega z^2(x,t)\,dx\,dt = \int_0^T\int_\Omega p(x,t)\,a(x,t)\,dx\,dt + \int_0^T\Big(\int_\Omega p(x,t)\overline{C}(x)\,dx\Big)u(t)\,dt.$$

We can write $J^0(u)$ in the form

$$J^0(u) = \int_0^T u^2(t)\,dt - \int_0^T G(t)u(t)\,dt + \int_0^T\int_\Omega p(x,t)a(x,t)\,dx\,dt,$$

where $G(t) := \int_\Omega \overline{C}(x)p(x,t)\,dx$.

From the stationarity conditions with respect to $u$, we obtain

$$DJ^0(u) = 2\big(u - G(t)\big) = 0 \implies u^0(t) = G(t) = \int_\Omega \overline{C}(x)\,p(x,t)\,dx.$$

Consequently,

$$\sigma_0 = J^0(u^0). \tag{3.12}$$

# 4 Main results

First, we verify that problem (2.1)–(2.6) admits a solution.

**Theorem 4.1.** *Let $f_0 \in L^2(\Omega)$, $f_1 \in L^2(Q_T)$, $f_2 \in L^2(0,T)$. Assume, moreover, that problem (2.1) has a unique solution in the space $W(0,T)$. Then, for every $\varepsilon > 0$, there exists a minimax estimate of functional (2.4) of form (2.5), and a function $u^\varepsilon \in L^2(0,T)$ that minimizes the minimax estimation error (2.7).*

*Proof.* For the fixed $u \in L^2(0,T)$, consider $g_1(\varphi) := l(\varphi) - \widehat{\widehat{l}}(\varphi)$. Arguing as in Section 3 for the averaged problem, from (3.7) we obtain that $g_1(\varphi)$ has the form

$$g_1(\varphi) = -\int_0^T u(t)f_2(t)\,dt + \int_\Omega z^\varepsilon(x,0)\,f_0(x)\,dx + \int_0^T\int_\Omega z^\varepsilon(x,t)\,f_1(x,t)\,dx\,dt,$$

where $z^\varepsilon$ solves

$$\begin{cases} \dfrac{\partial z^\varepsilon}{\partial t} - Az^\varepsilon = -a(x,t) + C\Big(x, \dfrac{t}{\varepsilon}\Big)u(t), \\ z^\varepsilon\big|_{t=T} = 0, \\ z^\varepsilon\big|_{\partial\Omega} = 0. \end{cases} \tag{4.1}$$

Taking into account Remark 1, by similar suggestions from [8] applied to (4.1), we find that problem (4.1) has a unique solution.

Since $z^\varepsilon$ does not depend on $f_0$, $f_1$, formulas (3.8)–(3.10) remain valid for $\varepsilon > 0$. Hence, $f_0^\varepsilon = \lambda^\varepsilon z^\varepsilon(x,0)$, $f_1^\varepsilon = \lambda^\varepsilon z^\varepsilon$, $f_2^\varepsilon = -\lambda^\varepsilon u$, with $f_0^\varepsilon$, $f_1^\varepsilon$, $f_2^\varepsilon$ satisfying (2.3), and $(\lambda^\varepsilon)^2 = (\|u\|^2_{L^2(0,T)} + \|z^\varepsilon(\,\cdot\,,0)\|^2_{L^2(\Omega)} + \|z^\varepsilon\|^2_{L^2(Q_T)})^{-1}$. Therefore, $J_\varepsilon(u)$ has the explicit representation

$$J_\varepsilon(u) = \|z^\varepsilon(\,\cdot\,,0)\|^2_{L^2(\Omega)} + \|z^\varepsilon\|^2_{L^2(Q_T)} + \|u\|^2_{L^2(0,T)}. \tag{4.2}$$

Next, from the first-order optimality conditions, analogously to the averaged problem, for (2.6) and every $\varepsilon > 0$, we obtain for $u^\varepsilon$ the relation

$$u^\varepsilon(t) = \int_\Omega C\Big(x, \frac{t}{\varepsilon}\Big)p^\varepsilon(x,t)\,dx, \tag{4.3}$$

where $p^\varepsilon$ solves

$$\begin{cases} \dfrac{\partial p^\varepsilon}{\partial t} + Ap^\varepsilon = z^\varepsilon, \\ p^\varepsilon(x,0) = z^\varepsilon(x,0), \\ p^\varepsilon\big|_{\partial\Omega} = 0, \end{cases} \tag{4.4}$$

and $z^\varepsilon$ is the solution of (4.1) with the control $u^\varepsilon$ given by (4.3).

Thus, the error (2.7) takes the form

$$\sigma_\varepsilon = J_\varepsilon(u^\varepsilon) = \|u^\varepsilon\|^2_{L^2(0,T)} + \|z^\varepsilon(\,\cdot\,,0)\|^2_{L^2(\Omega)} + \|z^\varepsilon\|^2_{L^2(Q_T)}. \tag{4.5}$$

□

We construct an approximate minimax estimate for the functional (2.4).

Let us present the following well-known auxiliary result, which will be used in the proof of the theorem below.

**Lemma 4.1** ([10]). *If for the parameters of the problem*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + A\varphi = f_n, \\ \varphi\big|_{t=0} = \varphi^n, \\ \varphi\big|_{\partial\Omega} = 0, \end{cases} \tag{4.6}$$

*we have that $f_n \to f$ weakly in $L^2(Q_T)$ and $\varphi_0^n \to \varphi_0$ weakly in $L^2(\Omega)$, then $\varphi^n \to \varphi$ in $L^2(Q_T)$, $\varphi^n \to \varphi$ in $C([\delta,T];L^2(\Omega))$ for all $\delta$. If $\varphi_0^n \to \varphi_0$ strongly in $L^2(\Omega)$, then $\varphi^n \to \varphi$ in $C([0,T];L^2(\Omega))$, where $\varphi$ is the solution of (4.6) with the right-hand side $f$ and initial data $\varphi_0$.*

**Theorem 4.2.** *The estimate*

$$\widehat{l}(\varphi) = \int_0^T u^0(t) y^\varepsilon(t)\,dt \tag{4.7}$$

*is an approximate minimax estimate for problem (2.1)–-(2.6). The errors satisfy*

$$\sigma_\varepsilon \to \sigma_0 \ \text{ as } \ \varepsilon \to 0, \tag{4.8}$$

$$\widehat{\sigma}_\varepsilon \to \sigma_0 \ \text{ as } \ \varepsilon \to 0, \tag{4.9}$$

*where $\sigma_\varepsilon$ and $\sigma_0$ are given by (4.5) and (3.12), respectively, and*

$$\widehat{\sigma}_\varepsilon = J_\varepsilon(u^0). \tag{4.10}$$

*For a sufficiently small $\varepsilon > 0$, the following holds: for every $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$,*

$$|\sigma_\varepsilon - \widehat{\sigma}_\varepsilon| < \eta, \tag{4.11}$$

*i.e., errors (4.5) and (4.10) are close for all sufficiently small $\varepsilon > 0$.*

*Proof.* It is known from Theorem 4.1 that for every $\varepsilon > 0$, there exists a unique $u^\varepsilon \in L^2(0,T)$ minimizing the functional $J^\varepsilon$, which has the explicit form (4.2). Moreover, $u^\varepsilon$ is given by relation (4.3).

Next, we use the arguments and calculations from Theorem 4.1. To verify that (4.7) is an approximate minimax estimate, we first pass to the limit in problem (4.4), taking into account the dependence of $p^\varepsilon$ on $z^\varepsilon$, where $z^\varepsilon$ is the solution of problem (4.1).

Consider the relation

$$\|u^\varepsilon\|^2_{L^2(0,T)} \le J_\varepsilon(u^\varepsilon) \le J_\varepsilon(0) = \|\widetilde{z}^\varepsilon(x,0)\|^2_{L^2(\Omega)} + \|\widetilde{z}^\varepsilon\|^2_{L^2(Q_T)},$$

where $\widetilde{z}^\varepsilon$ is the solution of

$$\begin{cases} \dfrac{\partial \widetilde{z}^\varepsilon}{\partial t} - A\widetilde{z}^\varepsilon = -a(x,t), \\ \widetilde{z}^\varepsilon(x,T) = 0, \\ \widetilde{z}^\varepsilon\big|_{\partial\Omega} = 0, \end{cases}$$

and, in particular, from (4.5) we find that there exists $C > 0$ such that $\|u^\varepsilon\|_{L^2(0,T)} \le C$, and along a subsequence $u^\varepsilon \to v$ weakly in $L^2(0,T)$.

Furthermore, due to the assumptions on the function $C(x, \frac{t}{\varepsilon})$ and the properties of the sequence $u^\varepsilon$, the sequence $\{C(x, \frac{t}{\varepsilon})u^\varepsilon(t)\}$ is bounded in $L^2(Q_T)$. Hence, for some $\chi \in L^2(Q_T)$, we have $C(x, \frac{t}{\varepsilon})u^\varepsilon(t) \to \chi$ weakly in $L^2(Q_T)$. Then, by Lemma 4.1, we obtain $z^\varepsilon \to \overline{z}$ in $C([0,T]; L^2(\Omega))$, where $\overline{z}$ is the solution of (4.1) with the right-hand side $-a(x,t) + \chi$.

Applying Lemma 4.1 to problem (4.4), we get

$$p^\varepsilon \to \overline{p} \ \text{ in } C([0,T]; L^2(\Omega)), \tag{4.12}$$

where $\overline{p}$ is the solution of (4.4) with the right-hand side $\overline{z}$.

Now, we pass to the limit in (4.3). Let $C_\varepsilon(x,t) := C(x, \frac{t}{\varepsilon})$. By the assumptions on $C(x, \frac{t}{\varepsilon})$, we have

$$\int_\Omega \bigg( \int_0^T |C_\varepsilon(x,s) - \overline{C}(x)|\, ds \bigg)^2 dx \to 0, \ \ T \to \infty. \tag{4.13}$$

From (4.13) we have that $C_\varepsilon(x,s) \to \overline{C}(x)$ in $L^1(\Omega \times (0,T))$, therefore up to a subsequence we have

$$C_\varepsilon(x,s) \to \overline{C}(x) \ \text{ a.e. on } \Omega \times (0,T), \ \ \varepsilon \to 0. \tag{4.14}$$

From (4.12), (4.14) and definition of function $C_\varepsilon(x,t)$ we obtain, up to a subsequence, that

$$C\Big(x, \frac{t}{\varepsilon}\Big)p^\varepsilon(x,t) \longrightarrow \overline{C}(x)\overline{p}(t,x) \ \text{ a.e.}$$

Then, by the Lebesgue theorem, for almost all $t$, we obtain

$$u^\varepsilon(t) = \int_\Omega C\Big(x, \frac{t}{\varepsilon}\Big)p^\varepsilon(x,t)\, dx \longrightarrow \int_\Omega \overline{C}(x)\overline{p}(t,x)\, dx \ \text{ a.e.}$$

And in view of the weak convergence $u^\varepsilon \to v$, we conclude that $v(t) = \int_\Omega \overline{C}(x)\overline{p}(t,x)\, dx$, and

$$u^\varepsilon(t) \to v(t) \ \text{ a.e.} \implies (u^\varepsilon)^2(t) \to v^2(t) \ \text{ a.e.} \tag{4.15}$$

Moreover there exists $L > 0$ such that $|u^\varepsilon|^2 \le L$. Then in view of Lebesgue's theorem we have that

$$\int_0^T |u^\varepsilon|^2\, dt \longrightarrow \int_0^T v^2\, dt, \ \ \varepsilon \to 0, \tag{4.16}$$

and taking into account strong convergence criterion from (4.16) and weak convergence $u^\varepsilon \to v$, $\varepsilon \to 0$, we obtain $u^\varepsilon \to v$, $\varepsilon \to 0$ in $L^2(0,T)$.

Moreover, $\chi = \overline{C}(x)v$.

In view of the explicit form of functionals (3.10), (4.2), we obtain (4.8), (4.9), and therefore (4.11), as well. □


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