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THE INITIAL VALUE PROBLEM FOR A HIGHER ORDER DIFFERENTIAL EQUATION AND THE CONFLUENT VANDERMONDE MATRIX

Abstract. An explicit formula for solving the initial value problem for a general linear nonhomogeneous differential equation with constant coefficients is given. This formula is obtained using the Vandermonde matrix or the confluent Vandermonde matrix constructed from the roots of the characteristic equation associated with the differential equation.

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რეზიუმე. მოცემულია ცხადი სახის ფორმულა მუდმივკოეფიციენტებიანი ზოგადი წრფივი არაერთგვაროვანი დიფერენციალური განტოლებისათვის საწყისი ამოცანის ამონახსნელად. ეს ფორმულა მიიღება ვანდერმონდის მატრიცის ან ვანდერმონდის კონფლუენტური მატრიცის გამოყენებით, რომელიც აგებულია დიფერენციალურ განტოლებასთან დაკავშირებული მახასიათებელი განტოლების ფესვების საშუალებით.

1 Introduction

It is well known that the Vandermonde matrix has numerous applications in various fields of mathematics [1,2,7]. One such application is to construct a solution to the initial value problem for a linear nonhomogeneous differential equation with constant coefficients. This application is formulated as follows (see Theorem 3.5 in [3]).

Let us fix $p \geq 2$, a finite or infinite interval I, and consider the differential equation

$$x^{(p)}(t) = a_1 x^{(p-1)}(t) + \cdots + a_{p-1}x'(t) + a_p x(t) + y(t), \quad t \in I, \tag{1.1}$$

in which $a_1, a_2, \ldots, a_p$ are fixed complex numbers, $y : I \to \mathbb{C}$ is some function continuous on I. If the characteristic equation, associated to (1.1),

$$Q(\lambda) \equiv \lambda^p - a_1\lambda^{p-1} - \cdots - a_{p-1}\lambda - a_p = 0 \tag{1.2}$$

has simple roots $\lambda_1, \lambda_2, \ldots, \lambda_p$, then the solution to the initial value problem for the differential equation (1.1) with the initial conditions

$$x(t_0) = x_0,\ x'(t_0) = x_1,\ \ldots,\ x^{(p-1)}(t_0) = x_{p-1}, \tag{1.3}$$

where $t_0 \in I$ and $x_0, x_1, \ldots x_{p-1}$ are fixed numbers, has the form

$$x(t) = \sum_{k=1}^{p} \left(e^{\lambda_k(t-t_0)}(L_{k1}x_0 + L_{k2}x_1 + \cdots + L_{kp}x_{p-1}) + \int_{t_0}^{t} e^{\lambda_k(t-s)} L_{kp} f(s)\, ds \right), \quad t \in I, \tag{1.4}$$

where L_{kj}, $1 \leq k, j \leq p$, are the elements of the matrix W^{-1}, inverse to the Vandermonde matrix

$$W = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_p \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_p^2 \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{p-1} & \lambda_2^{p-1} & \ldots & \lambda_p^{p-1} \end{pmatrix}.$$

The purpose of this paper is to obtain a generalization of formula (1.4) to the case where the characteristic equation (1.2) does not necessarily have simple roots.

2 Confluent Vandermonde matrices

Let the function $f : \mathbb{C} \to \mathbb{C}^p$ be defined by the formula

$$f(z) = col(1, z, \ldots, z^{p-1}).$$

Note that for each $1 \leq k \leq p-1$, there is

$$\frac{f^{(k)}(z)}{k!} = col\left(0, \ldots, 0, 1, \binom{k+1}{k} z, \binom{k+2}{k} z^2, \ldots, \binom{p-1}{k} z^{p-1-k}\right).$$

To each set of natural numbers q, $n_1, n_2, \ldots, n_q$, such that $1 \leq q \leq p$, $n_1 + n_2 + \cdots + n_q = p$, and to the set of complex numbers $z_1, z_2, \ldots, z_q$, we associate the confluent Vandermonde matrix

$$V(z_1, \ldots, z_q; n_1, \ldots, n_q) = \left(f(z_1)\ \frac{f'(z_1)}{1!} \cdots \frac{f^{(n_1-1)}(z_1)}{(n_1-1)!} \quad \cdots \quad f(z_q)\ \frac{f'(z_q)}{1!} \cdots \frac{f^{(n_q-1)}(z_q)}{(n_q-1)!} \right).$$

In particular,

$$V(z_1,z_2,z_3;2,1,3)=\begin{pmatrix}1&0&1&1&0&0\\ z_1&1&z_2&z_3&1&0\\ z_1^2&2z_1&z_2^2&z_3^2&2z_3&1\\ z_1^3&3z_1^2&z_2^3&z_3^3&3z_3^2&3z_3\\ z_1^4&4z_1^3&z_2^4&z_3^4&4z_3^3&6z_3^2\\ z_1^5&5z_1^4&z_2^5&z_3^5&5z_3^4&10z_3^3\end{pmatrix}.$$

It is known (see, e.g., [6], [5, App. A. 16]) that for $q\geq 2$ the confluent Vandermonde determinant $D=\det(V(z_1,\dots,z_q;n_1,\dots,n_q))$ is calculated by the formula

$$D=\prod_{1\leq i<j\leq q}(z_j-z_i)^{n_in_j}. \tag{2.1}$$

Let us put

$$T_A=\begin{pmatrix}0&1&0&\cdots&0&0\\ 0&0&1&\cdots&0&0\\ \hdotsfor{6}\\ 0&0&0&\cdots&0&1\\ a_p&a_{p-1}&a_{p-2}&\cdots&a_2&a_1\end{pmatrix},$$

where $a_1,a_2,\dots,a_p$ are the coefficients of equation (1.1), and denote by $J(\lambda,m)$ the $m\times m$ Jordan block with an eigenvalue λ.

The following theorem contains the property of the confluent Vandermonde matrix associated with the differential equation (1.1) that will be used later.

Theorem 2.1. *If $\lambda_1,\lambda_2,\dots,\lambda_q$ are the pairwise distinct roots of equation (1.2) of multiplicity $m_1,m_2,\dots,m_q$, respectively, with $m_1+m_2+\cdots+m_q=p$, that is, the polynomial $Q(\lambda)$ from (1.2) can be factored as*

$$Q(\lambda)=(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_q)^{m_q},$$

then the matrix $V=V(\lambda_1,\dots,\lambda_q;m_1,\dots,m_q)$ is invertible and

$$T_A=VJV^{-1}, \tag{2.2}$$

where

$$J=J(\lambda_1,\dots,\lambda_q;m_1,\dots,m_q)=J(\lambda_1,m_1)\oplus J(\lambda_2,m_2)\oplus\cdots\oplus J(\lambda_q,m_q)$$

is the Jordan normal form of the matrix T_A with Jordan blocks $J(\lambda_1,m_1),J(\lambda_2,m_2),\dots,J(\lambda_q,m_q)$ on the main diagonal.

Proof. In the case where $q=1$, the matrix

$$V(\lambda_1,p)=\begin{pmatrix}1&0&\cdots&0\\ \lambda_1&1&\cdots&0\\ \lambda_1^2&2\lambda_1&\cdots&0\\ \hdotsfor{4}\\ \lambda_1^{p-1}&(p-1)\lambda_1^{p-2}&\cdots&1\end{pmatrix}$$

is obviously invertible, and to prove equality (2.2), it is enough to verify that

$$T_AV(\lambda_1,p)=V(\lambda_1,p)J(\lambda_1,p). \tag{2.3}$$

It is directly verified that

$$T_A V(\lambda_1,p)=\begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & 0 & 0\\ \lambda_1^2 & 2\lambda_1 & 1 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \lambda_1^{p-1} & \binom{p-1}{1}\lambda_1^{p-2} & \binom{p-1}{2}\lambda_1^{p-3} & \cdots & \binom{p-1}{p-2}\lambda_1 & 1\\ A(\lambda_1) & \dfrac{A'(\lambda_1)}{1!} & \dfrac{A''(\lambda_1)}{2!} & \cdots & \dfrac{A^{(p-2)}(\lambda_1)}{(p-2)!} & \dfrac{A^{(p-1)}(\lambda_1)}{(p-1)!} \end{pmatrix},$$

where

$$A(\lambda)=a_1\lambda^{p-1}+a_2\lambda^{p-2}+\cdots+a_{p-1}\lambda+a_p,\quad \lambda\in\mathbb{C}.$$

Since λ_1 is the root of equation (1.2) of multiplicity p, then

$$\frac{A^k(\lambda_1)}{k!}=\binom{p}{k}\lambda_1^{p-k}.$$

Therefore, having calculated the product of the matrices $V(\lambda_1,p)J(\lambda_1,p)$, we conclude that equality (2.2) is satisfied.

If $q\ge 2$, then the invertibility of the matrix V follows from equality (2.1), and the equality $T_AV=VJ$ is checked in the same way as equality (2.3). □

Remark. If the condition of Theorem 2.1 is satisfied, then for each $1\le k\le q$, the vector $\frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!}$ is a generalized eigenvector of rank m_k corresponding to the matrix T_A and the eigenvalue λ_k, and the Jordan chain generated by $\frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!}$ has the form

$$\Big\{\frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!},\frac{f^{(m_k-2)}(\lambda_k)}{(m_k-2)!},\dots,\frac{f'(\lambda_k)}{1!},f(\lambda_k)\Big\}.$$

For determining the inverse matrix of a confluent Vandermonde matrix, see, e.g., [2,4].

3 Solution to the initial value problem

The following theorem shows how one can obtain a solution to the initial value problem (1.1), (1.3) using the related confluent Vandermonde matrix.

Theorem 3.1. *If $\lambda_1,\lambda_2,\dots,\lambda_q$ are pairwise distinct roots of equation (1.2) of multiplicity $m_1,m_2,\dots,m_q$, respectively, with $m_1+m_2+\cdots+m_q=p$, then the solution to the initial value problem (1.1), (1.3) is represented as*

$$\begin{aligned} x(t)=&\sum_{k=1}^{m_1}\Big(\frac{(t-t_0)^{k-1}}{(k-1)!}e^{\lambda_1(t-t_0)}\sum_{j=1}^{p}R_{kj}x_{j-1}+\int_{t_0}^{t}\frac{(t-s)^{k-1}}{(k-1)!}e^{\lambda_1(t-s)}R_{kp}y(s)\,ds\Big)\\ &+\sum_{k=1}^{m_2}\Big(\frac{(t-t_0)^{k-1}}{(k-1)!}e^{\lambda_2(t-t_0)}\sum_{j=1}^{p}R_{(m_1+k)j}x_{j-1}+\int_{t_0}^{t}\frac{(t-s)^{k-1}}{(k-1)!}e^{\lambda_2(t-s)}R_{(m_1+k)p}y(s)\,ds\Big)+\cdots\\ &+\sum_{k=1}^{m_q}\Big(\frac{(t-t_0)^{k-1}}{(k-1)!}e^{\lambda_q(t-t_0)}\sum_{j=1}^{p}R_{(m_1+\cdots+m_{q-1}+k)j}x_{j-1}\\ &\qquad+\int_{t_0}^{t}\frac{(t-s)^{k-1}}{(k-1)!}e^{\lambda_q(t-s)}R_{(m_1+\cdots+m_{q-1}+k)p}y(s)\,ds\Big), \end{aligned}\tag{3.1}$$

where R_{ij}, $1\le i,j\le p$, are the elements of the matrix V^{-1}, inverse of the confluent Vandermonde matrix $V=V(\lambda_1,\dots,\lambda_q;m_1,\dots,m_q)$.

Proof. Note that the initial value problem (1.1), (1.3) is equivalent to the matrix initial value problem

$$\begin{pmatrix} x_1(t) \\ \cdots \\ x_{p-1}(t) \\ x_p(t) \end{pmatrix}' = T_A \begin{pmatrix} x_1(t) \\ \cdots \\ x_{p-1}(t) \\ x_p(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \cdots \\ 0 \\ y(t) \end{pmatrix}, \quad t \in I, \tag{3.2}$$

$$\begin{pmatrix} x_1(t_0) \\ \cdots \\ x_{p-1}(t_0) \\ x_p(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ \cdots \\ x_{p-2} \\ x_{p-1} \end{pmatrix}, \tag{3.3}$$

which is considered in $\mathbb{C}^p$, and the vector function

$$\overline{x}(t) = col\big(x_1(t), \ldots, x_{p-1}(t), x_p(t)\big), \ \ t \in I,$$

is a solution to the initial value problem (3.2), (3.3) if and only if the function $x_1(t)$, $t \in I$, is a solution to the initial value problem (1.1), (1.3).

According to Theorem 2.1, $T_A = VJV^{-1}$. Therefore, multiplying equalities (3.2), (3.3) by the matrix V^{-1} on the left and making the substitution $\overline{u}(t) = V^{-1}\overline{x}(t)$, we obtain the equivalent to (3.2), (3.3) initial value problem

$$\overline{u}'(t) = J\overline{u}(t) + \overline{v}(t), \ \ t \in I, \tag{3.4}$$

$$\overline{u}(t_0) = \overline{u}_0, \tag{3.5}$$

in which

$$\overline{v}(t) = V^{-1}col(0, \ldots, 0, y(t)) = col\big(R_{1p}y(t), \ldots, R_{(p-1)p}y(t), R_{pp}y(t)\big), \ \ t \in I,$$

$$\overline{u}_0 = V^{-1}col(x_0, \ldots, x_{p-2}, x_{p-1}) = col\Big(\sum_{k=1}^{p} R_{1k}x_{k-1}, \ldots, \sum_{k=1}^{p} R_{(p-1)k}x_{k-1}, \sum_{k=1}^{p} R_{pk}x_{k-1}\Big).$$

The solution to the initial value problem (3.4), (3.5) has the form

$$\overline{u}(t) = e^{J(t-t_0)}\overline{u}_0 + \int_{t_0}^{t} e^{J(t-s)}\overline{v}(s)\, ds, \ \ t \in I,$$

where

$$e^{Jt} = e^{J(\lambda_1,m_1)t} \oplus e^{J(\lambda_2,m_2)t} \oplus \cdots \oplus e^{J(\lambda_q,m_q)t}$$

and

$$e^{J(\lambda_k,m_k)t} = \begin{pmatrix} e^{\lambda_k t} & \frac{t}{1!}\, e^{\lambda_k t} & \frac{t^2}{2!}\, e^{\lambda_k t} & \cdots & \frac{t^{m_k-1}}{(m_k-1)!}\, e^{\lambda_k t} \\ 0 & e^{\lambda_k t} & \frac{t}{1!}\, e^{\lambda_k t} & \cdots & \frac{t^{m_k-2}}{(m_k-2)!}\, e^{\lambda_k t} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_k t} \end{pmatrix}, \ \ 1 \le k \le q.$$

Therefore, the coordinates $u_1(t), u_{m_1+1}(t), \ldots, u_{m_1+\cdots+m_{q-1}+1}(t)$ of the vector function $\overline{u}(t)$ are determined by the formulas

$$u_{l_i+1}(t) = \sum_{k=1}^{m_i} \bigg(\frac{(t-t_0)^{k-1}}{(k-1)!}\, e^{\lambda_i(t-t_0)} \sum_{j=1}^{p} R_{(l_i+k)j}x_{j-1} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!}\, e^{\lambda_i(t-s)} R_{(l_i+k)p}y(s)\, ds \bigg), \ \ t \in I, \ \ 1 \le i \le q, \tag{3.6}$$

where $l_1 = 0$, $l_i = m_1 + \cdots + m_{i-1}$, $2 \le i \le q$. Since the elements of the first row of the matrix V have the form

$$V_{1j} = \begin{cases} 1, & j \in \{l_i + 1 : \ 1 \le i \le q\}, \\ 0, & \text{otherwise}, \end{cases}$$

and the solution to the initial value problem (1.1), (1.3) coincides with the first coordinate of the vector function $\overline{x}(t) = V\overline{u}(t)$, $t \in I$, then, by virtue of (3.6), this solution has the form (3.1). □

4 Example

Let us solve the initial value problem

$$x^{(4)} = 5x''' - 8x'' + 4x' + e^{2t}, \ \ t \in \mathbb{R}, \tag{4.1}$$

$$x(0) = x'(0) = 4, \quad x''(0) = x'''(0) = -4. \tag{4.2}$$

The corresponding to (4.1) characteristic equation $\lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda = 0$ has a root $\lambda_1 = 2$ of multiplicity 2 and two simple roots $\lambda_2 = 1$ and $\lambda_3 = 0$. Therefore,

$$V = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 2^2 & 2\cdot 2 & 1 & 0 \\ 2^3 & 3\cdot 2^2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 4 & 4 & 1 & 0 \\ 8 & 12 & 1 & 0 \end{pmatrix}, \quad V^{-1} = -\frac{1}{4}\begin{pmatrix} 0 & 8 & -11 & 3 \\ 0 & -4 & 6 & -2 \\ 0 & -16 & 16 & -4 \\ -4 & 8 & -5 & 1 \end{pmatrix}.$$

Thus, it follows from (3.1) that the solution to the initial value problem (4.1), (4.2) is

$$\begin{aligned} x(t) &= e^{2t}(-8 - 11 + 3) + te^{2t}(4 + 6 - 2) \\ &\quad + \int_0^t e^{2(t-s)} \cdot \Big(-\frac{3}{4}\Big) \cdot e^{2s}\, ds + \int_0^t (t-s)e^{2(t-s)} \cdot \frac{1}{2} \cdot e^{2s}\, ds + e^t(16 + 16 - 4) \\ &\quad + \int_0^t (t-s)e^{2(t-s)} \cdot 1 \cdot e^{2s}\, ds + e^{0t}(4 - 8 - 5 + 1) + \int_0^t e^{0(t-s)} \cdot \Big(-\frac{1}{4}\Big) \cdot e^{2s}\, ds \\ &= \frac{1}{4}t^2e^{2t} + \frac{29}{4}te^{2t} - \frac{121}{8}e^{2t} + 27e^t - \frac{63}{8}. \end{aligned}$$

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