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**A NONLOCAL IN TIME PROBLEM
FOR A SINGULAR PARABOLIC EQUATION**

Abstract. In this paper, we study a nonlocal multipoint in time problem for the evolution equation with the pseudo-Bessel operator. We also prove the well-posedness of the above-mentioned problem with an initial function that is an element of a certain space of generalized functions, investigate the properties of the fundamental solution of the problem, and find a representation of the solution in the form of a convolution of the fundamental solution with the initial function.

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1 Introduction

In this paper, we study a nonlocal multipoint in time problem in the half-space $t > 0$ for the evolution equation $\frac{\partial u}{\partial t} + Au = 0$ with the pseudo-Bessel operator $A = F_{B_\nu}^{-1}[|\sigma|^\alpha F_{B_\nu}]$, where $\alpha \in (1; +\infty) \setminus \{2, 3, 4, \dots\}$ is a fixed number (note that the pseudodifferential operator constructed using the symbol $|\sigma|^\alpha$, $\sigma \in \mathbb{R}$, by means of the Fourier transform, i.e., the operator $F^{-1}[|\sigma|^\alpha F]$, coincides with the restriction of the self-adjoint operator in $L_2(\mathbb{R})$ of the operator $|D|^\alpha$, $D = \frac{d}{dx}$, to a locally convex topological space that is the projective limit of certain Banach spaces continuously embedded in each other). The problem under study is a generalization of the Cauchy problem, where the initial condition $u(t, \cdot)|_{t=0} = f$ is replaced by the condition

$$\sum_{k=0}^m \mu_k u(t, \cdot)|_{t=t_k} = f,$$

where $t_0 = 0$, $\{t_1, \dots, t_m\} \subset (0; +\infty)$, $0 < t_1 < \dots < t_m < +\infty$, $\{\mu_0, \mu_1, \dots, \mu_m\} \subset \mathbb{R}$, $m \in \mathbb{N}$ are fixed numbers (if $\mu_0 = 1$, $\mu_1 = \dots = \mu_m = 0$, then we obviously have the Cauchy problem). This condition is interpreted in the classical sense or in the weak sense if f is a generalized function, that is, as the limiting relation

$$\sum_{k=0}^m \mu_k \lim_{t \rightarrow t_k} \langle u(t, \cdot), \varphi \rangle = \langle f, \varphi \rangle$$

for any function φ from the base space (here, $\langle f, \varphi \rangle$ denotes the action of the functional f on the test function). Such a nonlocal in time problem belongs to the class of multipoint problems for differential-operator equations (for a review of works devoted to nonlocal problems for differential-operator equations and partial differential equations, see, for example, [4]).

The Cauchy problem and the nonlocal multipoint in time problem for evolution equations with pseudo-Bessel operators in the case $t \in (0; T]$, $0 < T < +\infty$, in spaces of generalized functions of distribution type were studied in [2–5]. The methodology for studying the fundamental solution of the nonlocal in time problem, developed in these works, consists in representing such a solution as a series whose terms are fundamental solutions of the Cauchy problem for the corresponding equations. Such a representation complicates the possibility of explicit identification in the estimates of derivatives of the fundamental solution (with respect to the spatial variable) the dependence on the time parameter. In the present work, a different methodology for investigation the fundamental solution is used, which makes it possible to study the behavior of the solution as the time variable grows unboundedly (stabilization of the solution). We also prove the well-posedness of the above-mentioned problem with an initial function that is an element of a certain space of generalized functions, investigate the properties of the fundamental solution of the problem, and find a representation of the solution in the form of a convolution of the fundamental solution with the initial function.

2 Nonlocal in time problem

Consider the evolution equation

$$\frac{\partial u(t, x)}{\partial t} + Au(t, x) = 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R} \equiv \Omega, \quad (2.1)$$

where $A = F_{B_\nu}^{-1}[|\sigma|^\alpha F_{B_\nu}]$ is the pseudo-Bessel operator in the space Ψ_α [4].

By a solution of equation (2.1) we mean a function $u(t, x)$, $(t, x) \in \Omega$, which:

- 1) is continuously differentiable with respect to the variable t ;
- 2) $u(t, \cdot) \in \mathcal{D}(A) \equiv \Psi_\alpha$ for each $t > 0$;
- 3) $u(t, x)$, $(t, x) \in \Omega$, satisfies equation (2.1).

For equation (2.1), we pose a nonlocal multipoint in time problem: find a solution of equation (2.1) that satisfies the condition

$$\mu u(0, x) - \mu_1 u(t_1, x) - \cdots - \mu_m u(t_m, x) = f(x), \quad x \in \mathbb{R}, \quad f \in \Psi_\alpha, \quad (2.2)$$

where $u(0, x) = \lim_{t \rightarrow +0} u(t, x)$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, $\{\mu, \mu_1, \dots, \mu_m\} \subset (0, +\infty)$, $\{t_1, \dots, t_m\} \subset (0, +\infty)$ are fixed numbers, $0 < t_1 < t_2 < \cdots < t_m < +\infty$, $\mu > \sum_{k=1}^m \mu_k$.

We look for a solution of problem (2.1), (2.2) using the Bessel transform in the form $u(t, x) = F_{B_\nu}^{-1}[v(t, \sigma)]$. For the function $v : \Omega \rightarrow \mathbb{R}$, we obtain a problem with parameter σ :

$$\frac{dv(t, \sigma)}{dt} + |\sigma|^\alpha v(t, \sigma) = 0, \quad (t, \sigma) \in \Omega, \quad (2.3)$$

$$\mu v(0, \sigma) - \sum_{k=1}^m \mu_k v(t_k, \sigma) = \tilde{f}(\sigma), \quad \sigma \in \mathbb{R}, \quad (2.4)$$

where $\tilde{f}(\sigma) = F_{B_\nu}[f]$. The solution of problem (2.3), (2.4) is given by the formula

$$v(t, \sigma) = \tilde{f}(\sigma) \exp\{-t|\sigma|^\alpha\} \left(\mu - \sum_{k=1}^m \mu_k \exp\{-t_k|\sigma|^\alpha\} \right)^{-1}, \quad (t, \sigma) \in \Omega.$$

Thus, the solution of problem (2.1), (2.2) has the form

$$u(t, x) = c_\nu \int_0^\infty v(t, \sigma) j_\nu(\sigma x) \sigma^{2\nu+1} d\sigma, \quad (t, x) \in \Omega.$$

We introduce the notation

$$G(t, x) = F_{B_\nu}^{-1}[Q(t, \sigma)] = c_\nu \int_0^\infty Q(t, \sigma) j_\nu(\sigma x) \sigma^{2\nu+1} d\sigma,$$

where

$$Q(t, \sigma) = Q_1(t, \sigma) Q_2(\sigma),$$

$$Q_1(t, \sigma) = \exp\{-t|\sigma|^\alpha\}, \quad Q_2(\sigma) = \left(\mu - \sum_{k=1}^m \mu_k \exp\{-t_k|\sigma|^\alpha\} \right)^{-1}.$$

Then, reasoning formally, we obtain

$$u(t, x) = \int_0^\infty T_x^\xi G(t, x) f(\xi) \xi^{2\nu+1} d\xi = G(t, x) * f(x).$$

Indeed,

$$u(t, x) = c_\nu \int_0^\infty Q(t, \sigma) \left(\int_0^\infty f(\xi) j_\nu(\sigma \xi) \xi^{2\nu+1} d\xi \right) j_\nu(\sigma x) \sigma^{2\nu+1} d\sigma.$$

Since $j_\nu(\sigma \xi) j_\nu(\sigma x) = T_x^\xi j_\nu(\sigma x)$ [4], using (formally) Fubini's theorem, we obtain

$$u(t, x) = c_\nu \int_0^\infty \left(\int_0^\infty Q(t, \sigma) T_x^\xi j_\nu(\sigma x) \sigma^{2\nu+1} d\sigma \right) f(\xi) \xi^{2\nu+1} d\xi = \int_0^\infty \Lambda(t, \xi, x) f(\xi) \xi^{2\nu+1} d\xi,$$

where

$$\Lambda(t, \xi, x) := c_\nu \int_0^\infty Q(t, \sigma) T_x^\xi j_\nu(\sigma x) \sigma^{2\nu+1} d\sigma.$$

Using Fubini's theorem once more, we find that

$$\begin{aligned} \Lambda(t, \xi, x) &= c_\nu b_\nu \int_0^\infty Q(t, \sigma) \left(\int_0^\pi j_\nu(\sigma \sqrt{x^2 + \xi^2 - 2x\xi \cos \omega}) \sin^{2\nu} \omega d\omega \right) \sigma^{2\nu+1} d\sigma \\ &= c_\nu b_\nu \int_0^\pi J(t, \xi, x) \sin^{2\nu} \omega d\omega, \end{aligned}$$

where

$$J(t, \xi, \omega) = \int_0^\infty Q(t, \sigma) j_\nu(\sigma r_{x,\xi}(\omega)) \sigma^{2\nu+1} d\sigma, \quad r_{x,\xi}(\omega) := \sqrt{x^2 + \xi^2 - 2x\xi \cos \omega}.$$

On the other hand,

$$\begin{aligned} T_x^\xi G(t, x) &= c_\nu b_\nu \int_0^\pi G(t, r_{x,\xi}(\omega)) \sin^{2\nu} \omega d\omega \\ &= c_\nu b_\nu \int_0^\pi \left(\int_0^\infty Q(t, \sigma) j_\nu(\sigma r_{x,\xi}(\omega)) \sigma^{2\nu+1} d\sigma \right) \sin^{2\nu} \omega d\omega \\ &= c_\nu b_\nu \int_0^\pi J(t, \xi, \omega) \sin^{2\nu} \omega d\omega = \Lambda(t, \xi, x). \end{aligned}$$

Therefore,

$$u(t, x) = \int_0^\infty T_x^\xi G(t, x) f(\xi) \xi^{2\nu+1} d\xi = G(t, x) * f(x).$$

To justify the transformations carried out here, we study the properties of the function $Q(t, \sigma)$ as a function of the variable σ , as well as the properties of the function G , since $G = F_{B_\nu}^{-1}[Q]$; the properties of the function $G(t, x)$ will, in turn, determine the properties of the function $u(t, x)$.

Lemma 2.1. *For each fixed $t > 0$, the function $Q(t, \sigma)$ is infinitely differentiable with respect to the variable $\sigma \in \mathbb{R} \setminus \{0\}$; for its derivatives, the following estimates hold*

$$|D_\sigma^s Q(t, \sigma)| \leq b_s t^\gamma |\sigma|^{\omega_s - s} \exp\{-t|\sigma|^\alpha\}, \quad \sigma \neq 0, \quad s \in \mathbb{N}, \quad (2.5)$$

where the constant $b_s = b_s(\alpha) > 0$ does not depend on t ,

$$\gamma = \begin{cases} 0 & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t > 1, \end{cases} \quad \omega_s = \begin{cases} \alpha & \text{if } \sigma \neq 0, \quad |\sigma| < 1, \\ \alpha s & \text{if } |\sigma| \geq 1. \end{cases}$$

Proof. To prove the statement, we use the formula

$$D_\sigma^k F(g(\sigma)) = \sum_{\tilde{m}=1}^k \frac{d^{\tilde{m}} F(g)}{dg^{\tilde{m}}} \sum \frac{k!}{\tilde{m}_1! \dots \tilde{m}_l!} \left(\frac{d}{d\sigma} g(\sigma) \right)^{\tilde{m}_1} \dots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} g(\sigma) \right)^{\tilde{m}_l}, \quad (2.6)$$

in which we set $F = e^g$, $g = -t|\sigma|^\alpha$. Then

$$|D_\sigma^s \exp\{-t|\sigma|^\alpha\}| \leq e^{-t|\sigma|^\alpha} \sum_{\tilde{m}=1}^s \sum \frac{s!}{\tilde{m}_1! \dots \tilde{m}_l!} \tilde{\Lambda}, \quad \sigma \neq 0, \quad s \in \mathbb{N},$$

where

$$\tilde{\Lambda} := \left| \left(\frac{d}{d\sigma} (-t|\sigma|^\alpha) \right)^{\tilde{m}_1} \left(\frac{1}{2!} \frac{d^2}{d\sigma^2} (-t|\sigma|^\alpha) \right)^{\tilde{m}_2} \cdots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} (-t|\sigma|^\alpha) \right)^{\tilde{m}_l} \right|, \quad \sigma \neq 0.$$

Since $\alpha > 1$, the following inequalities are valid:

$$\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(l-1)) \leq \alpha(\alpha+1)\cdots(\alpha+l) \leq \alpha \cdot 2\alpha \cdot 3\alpha \cdots l\alpha = \alpha^l l!.$$

Taking into account the last inequality, we find an estimate for $\tilde{\Lambda}$:

$$\begin{aligned} \tilde{\Lambda} &\leq t^{\tilde{m}_1} \alpha^{\tilde{m}_1} |\sigma|^{(\alpha-1)\tilde{m}_1} t^{\tilde{m}_2} \alpha^{2\tilde{m}_2} |\sigma|^{(\alpha-2)\tilde{m}_2} \cdots t^{\tilde{m}_l} \alpha^{l\tilde{m}_l} |\sigma|^{(\alpha-l)\tilde{m}_l} \\ &= t^{\tilde{m}_1+\cdots+\tilde{m}_l} \alpha^{\tilde{m}_1+2\tilde{m}_2+\cdots+l\tilde{m}_l} |\sigma|^{\alpha(\tilde{m}_1+\cdots+\tilde{m}_l)-(\tilde{m}_1+2\tilde{m}_2+\cdots+l\tilde{m}_l)} = t^{\tilde{m}} \alpha^s |\sigma|^{\alpha\tilde{m}-s}, \quad \sigma \neq 0. \end{aligned}$$

Thus,

$$\begin{aligned} |D_\sigma^s Q_1(t, \sigma)| &= |D_\sigma^s \exp\{-t|\sigma|^\alpha\}| \\ &\leq \alpha^s s! \sum_{\tilde{m}=1}^s t^{\tilde{m}} |\sigma|^{\alpha\tilde{m}-s} \exp\{-t|\sigma|^\alpha\} \leq c_s t^{\gamma s} |\sigma|^{\omega_s - s} \exp\{-t|\sigma|^\alpha\}, \quad \sigma \neq 0, \end{aligned} \quad (2.7)$$

where $c_s = c_s(\alpha) > 0$,

$$\gamma = \begin{cases} 0 & \text{if } 0 < t \leq 1, \\ 1 & \text{if } t > 1, \end{cases} \quad \omega_s = \begin{cases} \alpha & \text{if } \sigma \neq 0, |\sigma| < 1, \\ \alpha s & \text{if } |\sigma| \geq 1. \end{cases}$$

Taking into account the Leibniz formula for differentiating the product of two functions, we find that

$$D_\sigma^s Q(t, \sigma) = D_\sigma^s (Q_1(t, \sigma) Q_2(\sigma)) = \sum_{p=0}^s C_s^p Q_1^{(p)}(t, \sigma) \cdot Q_2^{(s-p)}(\sigma).$$

For further estimates, we use formula (2.6), in which we set $F = \varphi^{-1}$, $\varphi = R$, where

$$R(\sigma) = \mu - \sum_{k=1}^m \mu_k \exp\{-t_k |\sigma|^\alpha\} = Q_2^{-1}(\sigma).$$

Then $Q_2(\sigma) = F(\varphi) = R^{-1}$ and

$$|D_\sigma^s Q_2(\sigma)| = \left| \sum_{\tilde{m}=1}^s \frac{d^{\tilde{m}}}{dR^{\tilde{m}}} R^{-1} \sum_{\tilde{m}_1! \cdots \tilde{m}_l!} \frac{s!}{\tilde{m}_1! \cdots \tilde{m}_l!} \left(\frac{d}{d\sigma} R(\sigma) \right)^{\tilde{m}_1} \cdots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} R(\sigma) \right)^{\tilde{m}_l} \right|, \quad s \in \mathbb{N}, \quad \sigma \neq 0.$$

Let $|\sigma| \geq 1$. Taking into account the form of the function R and estimates (2.7), we obtain

$$\begin{aligned} \left| \frac{1}{j!} \frac{d^j}{d\sigma^j} R(\sigma) \right| &\leq \frac{1}{j!} \sum_{k=1}^m \mu_k \left| \frac{d^j}{d\sigma^j} e^{-t_k |\sigma|^\alpha} \right| \\ &\leq \frac{c_j}{j!} \sum_{k=1}^m \mu_k t_k^{\gamma j} |\sigma|^{\alpha j - j} e^{-t_k |\sigma|^\alpha} \leq c_j \sum_{k=1}^m \mu_k \frac{t_k^{\gamma j}}{t_k^j} |\sigma|^{-j} \equiv \beta_j |\sigma|^{-j}, \quad j \in \{1, \dots, l\} \end{aligned}$$

(here, we used the inequality $|\sigma|^{\alpha j} \exp\{-t_k |\sigma|^\alpha\} \leq \frac{j!}{t_k^j}$, $j \in \{1, \dots, l\}$).

Therefore, if $|\sigma| \geq 1$, then the following estimate is valid:

$$\begin{aligned} \Delta &:= \left| \left(\frac{d}{d\sigma} R(\sigma) \right)^{\tilde{m}_1} \cdots \left(\frac{1}{l!} \frac{d^l}{d\sigma^l} R(\sigma) \right)^{\tilde{m}_l} \right| \\ &\leq \beta_1^{\tilde{m}_1} |\sigma|^{-\tilde{m}_1} \beta_2^{\tilde{m}_2} |\sigma|^{-2\tilde{m}_2} \cdots \beta_l^{\tilde{m}_l} |\sigma|^{-l\tilde{m}_l} \leq \beta^{\tilde{m}_1+\cdots+\tilde{m}_l} |\sigma|^{-(\tilde{m}_1+2\tilde{m}_2+\cdots+l\tilde{m}_l)} = \beta^{\tilde{m}} |\sigma|^{-s}, \end{aligned}$$

where $\beta = \max\{\beta_1, \dots, \beta_l\}$.

If $\sigma \neq 0$, $|\sigma| < 1$, then

$$\left| \frac{1}{j!} \frac{d^j}{d\sigma^j} R(\sigma) \right| \leq c_j t_m^{\gamma_j} |\sigma|^{-j} \exp\{-t_1 |\sigma|^\alpha\} \leq \tilde{c}_j |\sigma|^{-j}, \quad \Delta \leq \tilde{\beta}^{\tilde{m}} |\sigma|^{-s}, \quad j \in \{1, \dots, l\}$$

(here, we took into account the fact that $0 < t_1 < t_2 < \dots < t_m < +\infty$). Moreover,

$$\frac{d^{\tilde{m}}}{dR^{\tilde{m}}} R^{-1} = (-1)^{\tilde{m}} \tilde{m}! R^{-(\tilde{m}+1)}.$$

Since $\exp\{-t_k |\sigma|^\alpha\} \leq 1$, $\forall \sigma \in \mathbb{R}$, $k \in \{1, \dots, m\}$, we have

$$\mu - \sum_{k=1}^m \mu_k \exp\{-t_k |\sigma|^\alpha\} \geq \mu - \sum_{k=1}^m \mu_k.$$

By assumption $\mu > \sum_{k=1}^m \mu_k$, therefore,

$$R^{-1}(\sigma) \leq \left(\mu - \sum_{k=1}^m \mu_k \right)^{-1} = \beta_0 > 0, \quad \left| \frac{d^{\tilde{m}}}{dR^{\tilde{m}}} R^{-1} \right| \leq \beta_0^{\tilde{m}+1} \tilde{m}!.$$

Taking into account all these inequalities, we obtain

$$|D_\sigma^s Q_2(\sigma)| \leq s! \sum_{\tilde{m}=1}^s \beta_0^{\tilde{m}+1} \tilde{\beta}^{\tilde{m}} \tilde{m}! |\sigma|^{-s} \equiv \tilde{c}_s |\sigma|^{-s}, \quad \sigma \neq 0, \quad (2.8)$$

where $\tilde{\beta} = \max\{\beta, \tilde{\beta}\}$. By virtue of (2.7), (2.8), we obtain the following estimate ($\forall \sigma \neq 0$):

$$\begin{aligned} |D_\sigma^s Q(t, \sigma)| &\leq \sum_{p=0}^s C_s^p t^{\gamma_p} |\sigma|^{\omega_p - p} \tilde{C}_{s-p} c_p |\sigma|^{-(s-p)} \exp\{-t |\sigma|^\alpha\} \\ &\leq b_s t^{\gamma_s} |\sigma|^{\omega_s - s} \exp\{-t |\sigma|^\alpha\}, \quad \sigma \neq 0, \quad s \in \mathbb{N}. \end{aligned} \quad \square$$

Remark 2.1. From estimates (2.5), (2.7) it follows that $\{Q_1(t, \cdot), Q(t, \cdot)\} \subset \Phi_\alpha$ for each $t > 0$. From this and the boundedness of the function Q_2 on \mathbb{R} , we obtain that Q_2 is a multiplier in the space Φ_α .

Remark 2.2. Reasoning similarly to the previous case, we find that for $t > 1$ the derivatives of the function

$$Q(t, t^{-1/\alpha} \sigma) = \exp\{-|\sigma|^\alpha\} \left(\mu - \sum_{k=1}^m \mu_k \exp\{-t^{-1} t_k |\sigma|^\alpha\} \right)^{-1}$$

satisfy the inequalities

$$|D_\sigma^s Q(t, t^{-1/\alpha} \sigma)| \leq L_s |\sigma|^{\omega_s - s} \exp\{-|\sigma|^\alpha\}, \quad \sigma \neq 0, \quad s \in \mathbb{N}, \quad (2.9)$$

where the constants $L_s > 0$ do not depend on t . If $0 < t \leq 1$, then

$$|D_\sigma^s Q(t, t^{-1/\alpha} \sigma)| \leq L'_s t^{-s} |\sigma|^{\omega_s - s} \exp\{-|\sigma|^\alpha\}, \quad \sigma \neq 0, \quad s \in \mathbb{N}. \quad (2.10)$$

We now study the properties of the function

$$G(t, x) = c_\nu \int_0^\infty Q(t, \sigma) j_\nu(\sigma x) \sigma^{2\nu+1} d\sigma, \quad (t, x) \in \Omega.$$

From Remark 2.1 it follows that for each $t > 0$, the function $G(t, x)$, as a function of x , is an element of the space $\Psi_\alpha = F_{B_\nu}[\Phi_\alpha]$. We isolate the dependence on the parameter t in the estimates of

the function G and its derivatives (with respect to the variable x). Let us introduce the notation $\nu = n + 1/2$, $n \in \mathbb{N}$, and use the representation of Bessel functions $J_{n+1/2}$ of half-integer order:

$$J_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \sin\left(x - \frac{n\pi}{2}\right) P_n\left(\frac{1}{x}\right) + \cos\left(x - \frac{n\pi}{2}\right) Q_n\left(\frac{1}{x}\right) \right\}, \quad x > 0,$$

where $P_n(\frac{1}{x})$ is a polynomial of degree n in $\frac{1}{x}$, $Q_n(\frac{1}{x})$ is a polynomial of degree $n - 1$; moreover, $P_n(0) = 1$, $Q_n(0) = 0$ (the Bessel function j_ν is a solution of Bessel's equation $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$). Since the normalized Bessel function j_ν is related to the Bessel function J_ν by the formula $j_\nu(x) = \frac{2^\nu \Gamma(\nu + 1/2)}{x^\nu} J_\nu(x)$, $x > 0$, we have the following representation for the function $j_{n+1/2}$:

$$j_{n+1/2}(x) = \frac{c_n}{x^{n+1}} \left\{ \sin\left(x - \frac{n\pi}{2}\right) P_n\left(\frac{1}{x}\right) + \cos\left(x - \frac{n\pi}{2}\right) Q_n\left(\frac{1}{x}\right) \right\}, \quad n \in \mathbb{N}, \quad x > 0.$$

For example,

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \sin(x - \pi) \left(1 - \frac{3}{x^2}\right) + \frac{3}{x} \cos(x - \pi) \right\}.$$

Lemma 2.2. *For the function $G(t, x)$, $(t, x) \in \Omega$, and its derivatives, the following estimates are valid:*

$$|D_x^m G(t, x)| \leq c_m t^{\lambda(m)} (t^{1/\alpha} + |x|)^{-(2\nu+2+[\alpha]+m)}, \quad m \in \mathbb{Z}_+, \quad (2.11)$$

where the constant $c_m > 0$ does not depend on $t > 0$,

$$\lambda(m) = \begin{cases} -\frac{(\alpha - 1)(m + \nu + 3/2) + \alpha[\alpha] + \nu + 1/2}{\alpha} & \text{if } 0 < t \leq 1, \\ \frac{[\alpha]}{\alpha} & \text{if } t > 1. \end{cases}$$

Proof. Consider the case $m = 0$. Making the change of integration variable $\sigma = t^{-1/\alpha}\xi$, for the function G , we obtain the following representation:

$$G(t, x) = t^{-(2\nu+2)/\alpha} G_0(t, z),$$

where

$$G_0(t, z) = c_\nu \int_0^\infty Q(t, t^{-1/\alpha}\xi) j_\nu(z\xi) \xi^{2\nu+1} d\xi, \quad z = t^{-1/\alpha}x.$$

Taking into account the form of the normalized Bessel function j_ν , $\nu \in \{3/2, 5/2, 7/2, \dots\}$, we find that $G(t, x) = t^{-(2\nu+2)/\alpha} (\Lambda_1(t, z) + \Lambda_2(t, z))$, where

$$\Lambda_1(t, z) = \frac{c_n}{z^{n+1}} \int_0^\infty Q(t, t^{-1/\alpha}\xi) \xi^{n+1} \sin\left(z\xi - \frac{n\pi}{2}\right) P_n\left(\frac{1}{z\xi}\right) d\xi,$$

$$\Lambda_2(t, z) = \frac{c_n}{z^{n+1}} \int_0^\infty Q(t, t^{-1/\alpha}\xi) \xi^{n+1} \cos\left(z\xi - \frac{n\pi}{2}\right) Q_n\left(\frac{1}{z\xi}\right) d\xi,$$

$$P_n\left(\frac{1}{z\xi}\right) = \sum_{k=0}^n \frac{b_k}{(z\xi)^k}, \quad Q_n\left(\frac{1}{z\xi}\right) = \sum_{k=0}^{n-1} \frac{d_k}{(z\xi)^k}, \quad \xi > 0.$$

Thus,

$$\begin{aligned}\Lambda_1(t, z) &= c_n \sum_{k=0}^n \frac{b_k}{z^{n+k+1}} \int_0^\infty Q(t, t^{-1/\alpha} \xi) \xi^{n-k+1} \sin\left(z\xi - \frac{n\pi}{2}\right) d\xi = c_n \sum_{k=0}^n \frac{b_k}{z^{n+k+1}} J_{1,k}(t, z), \\ J_{1,k}(t, z) &= \int_0^\infty Q(t, t^{-1/\alpha} \xi) \xi^{n-k+1} \sin\left(z\xi - \frac{n\pi}{2}\right) d\xi, \\ \Lambda_2(t, z) &= c_n \sum_{k=1}^{n-1} \frac{d_k}{z^{n+k+1}} J_{2,k}(t, z), \\ J_{2,k}(t, z) &= \int_0^\infty Q(t, t^{-1/\alpha} \xi) \xi^{n-k+1} \cos\left(z\xi - \frac{n\pi}{2}\right) d\xi.\end{aligned}$$

Since $z \neq 0$, integrating by parts $s = n - k + 2 + [\alpha]$ times, we represent $J_{1,k}(t, z)$ in the form

$$\begin{aligned}J_{1,k}(t, z) &= \lim_{\varepsilon \rightarrow +0} \int_\varepsilon^{+\infty} Q(t, t^{-1/\alpha} \xi) \xi^{n-k+1} \sin\left(z\xi - \frac{n\pi}{2}\right) d\xi \\ &= \frac{(-1)^s}{z^s} \lim_{\varepsilon \rightarrow +0} \left[\int_\varepsilon^{+\infty} D_\xi^s(Q(t, t^{-1/\alpha} \xi) \xi^{n-k+1}) \sin\left(z\xi - \frac{n\pi}{2} + s\frac{\pi}{2}\right) d\xi + r(\varepsilon, z) \right],\end{aligned}$$

where the symbol $r(\varepsilon, z)$ denotes the boundary term, which consists of terms of the form $cD_\xi^l \xi^{n-k+1} D_\xi^{s-1-l} Q(t, t^{-1/\alpha} \xi) \cdot \Lambda$ if $0 \leq l \leq n-1$, the term $cD_\xi^{s-1-l} Q(t, t^{-1/\alpha} \xi) \cdot \Lambda$ if $l = n-k+1$, and the term $cQ(t, t^{-1/\alpha} \xi) \xi^{n-k+1} \cdot \Lambda$ (c are the constants, whose specific values are not important at the moment), $\Lambda = \sin(z\xi - \frac{n\pi}{2} + (s-1)\frac{\pi}{2})$ with values at the points $\xi = \varepsilon$ and at infinity.

From estimates (2.9) it follows that for $0 < \xi < 1$ and $t > 1$ the following estimates hold: if $0 \leq l \leq n-k$, then

$$|D_\xi^l \xi^{n-k+1} D_\xi^{s-1-l} Q(t, t^{-1/\alpha} \xi)| \leq c \xi^{n-k+1-l} \xi^{\alpha-(s-1-l)} = \xi^{\alpha-[\alpha]} = \xi^{\{\alpha\}}, \quad s = n - k + 2 + [\alpha];$$

if $l = n - k + 1$, then

$$|D_\xi^{s-1-l} Q(t, t^{-1/\alpha} \xi)| \leq c \xi^{\alpha-(s-1-l)} = c \xi^{\alpha-[\alpha]} = c \xi^{\{\alpha\}}.$$

Moreover, $\lim_{\varepsilon \rightarrow +0} Q(t, t^{-1/\alpha} \varepsilon) \varepsilon^{n-k+1} = 0$, since

$$|Q(t, t^{-1/\alpha} \xi)| \leq \left(\mu - \sum_{k=1}^m \mu_k \right)^{-1}, \quad \forall t > 0, \quad \varepsilon \geq 0.$$

It follows from this that $\lim_{\varepsilon \rightarrow +0} r(\varepsilon, z) = 0$ for each $z \neq 0$. At infinity, the indicated terms vanish due to the decay to zero at infinity of the function $Q(t, t^{-1/\alpha} \xi)$ and its derivatives.

Taking into account the formula for differentiating the product of two functions, we obtain that the estimate of $|J_{1,k}|$ reduces to the estimate of a sum of integrals of the form

$$\begin{aligned}|J_{1,k}(t, z)| &\leq \frac{1}{|z|^s} \int_0^\infty |D_\xi^s(Q(t, t^{-1/\alpha} \xi) \cdot \xi^{n-k+1})| d\xi \\ &\leq \frac{1}{|z|^s} \left[\int_0^\infty |D_\xi^s Q(t, t^{-1/\alpha} \xi)| \xi^{n-k+1} d\xi + s(n-k+1) \int_0^\infty |D_\xi^{s-1} Q(t, t^{-1/\alpha} \xi)| \xi^{n-k} d\xi + \dots \right]\end{aligned}$$

$$\begin{aligned}
& + \frac{(n-k+1)!}{(n-k+1-j)!} C_s^{s-j} \int_0^\infty |D_s^{s-j} Q(t, t^{-1/\alpha} \xi)| \xi^{n-k+1-j} d\xi + \dots \\
& + (n-k+1)! \int_0^\infty |D_\xi^{s-(n-k+1)} Q(t, t^{-1/\alpha} \xi)| d\xi, \quad z \neq 0.
\end{aligned} \tag{2.12}$$

From the properties of the function $Q(t, t^{-1/\alpha} \xi)$ it follows that all integrals in (2.12) are convergent. Indeed, consider one of the terms in the sum (2.12), that corresponds to the index $n-k+1-j$ and clarify the behavior of the integrand in a sufficiently small (right-hand) neighborhood of the point $\xi \neq 0$ (at infinity, the function Q together with all its derivatives decays as $\exp\{-|\xi|^\alpha\}$). Taking into account (2.9), we find that

$$\xi^{n-k+1-j} |D_\xi^{s-j} Q(t, t^{-1/\alpha} \xi)| \leq L_{s-j} \xi^{\alpha-(s-j)} \xi^{n-k+1-j} = L_{s-j} \xi^{\alpha-[\alpha]-1} = L_{s-j} \xi^{\{\alpha\}-1}.$$

From this it follows that the indicated integrand has an integrable singularity at the point $\xi = 0$. Then

$$\begin{aligned}
& \int_0^\infty |D_\xi^{s-j} Q(t, t^{-1/\alpha} \xi)| \xi^{n-k+1-j} d\xi \\
& \leq L_{s-1} \left[\int_0^1 \xi^{\{\alpha\}-1} d\xi + \int_1^\infty \xi^{n-k+1-j} \xi^{\alpha(s-j)-(s-j)} \exp\{-\xi^\alpha\} d\xi \right] \\
& \leq L_{s-j} \left[\int_0^1 \xi^{\{\alpha\}-1} d\xi + \int_1^\infty \xi^{n+1+(\alpha-1)(n+2+[\alpha])} \exp\{-\xi^\alpha\} d\xi \right] < +\infty.
\end{aligned}$$

Therefore, if $z \neq 0$ ($z = t^{-1/\alpha} x$), $t > 1$, then the following inequality is valid:

$$|J_{1,k}(t, z)| \leq \beta_k |z|^{-s}, \quad s = n - k + 2 + [\alpha]. \tag{2.13}$$

Then, taking into account (2.13), we have

$$|\Lambda_1(t, z)| \leq c_n \sum_{k=0}^n |b_k| \beta_k |z|^{-(n+k+1)} |z|^{-(n-k+2+[\alpha])} = c'_n |z|^{-(2n+3+[\alpha])}.$$

We estimate $|\Lambda_2(t, z)|$ similarly. Thus, for $t > 1$ and $z \neq 0$, we have the estimate

$$|G_0(t, z)| \leq c |z|^{-(2n+3+[\alpha])}, \tag{2.14}$$

where the constant $c > 0$ does not depend on t .

Further, taking into account (2.9) and the Poisson integral representation for the normalized Bessel function j_ν , we note that for $t > 0$ and $z \in \mathbb{R}$,

$$|G_0(t, z)| \leq c_\nu \int_0^\infty |Q(t, t^{-1/\alpha} \xi)| |j_\nu(z\xi)| \xi^{2\nu+1} d\xi \leq c_\nu A_\nu \beta_0 \int_0^\infty \exp\{-\xi^\alpha\} d\xi = c_1, \tag{2.15}$$

where

$$c_\nu = (2^{2\nu} \Gamma^2(\nu+1))^{-1}, \quad A_\nu = \sqrt{\pi} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)}, \quad \beta_0 = \left(\mu - \sum_{k=1}^m \mu_k \right)^{-1}.$$

From (2.14), (2.15) it follows that the function $G_0(t, z)$ for $t > 1$ and $z \in \mathbb{R}$ satisfies the inequality

$$|G_0(t, z)| \leq \tilde{c} (1 + |z|)^{-(2n+3+[\alpha])} = \tilde{c} t^{(2n+3+[\alpha])/\alpha} (t^{1/\alpha} + |x|)^{-(2n+3+[\alpha])}, \quad x \in \mathbb{R}, \quad t > 1.$$

Indeed, we introduce the notation $\Gamma(t, z) = (1 + |z|)^{2n+3+[\alpha]}|G_0(t, z)|$, $z = t^{-1/\alpha}x$. If $|z| < 1$ ($|x| < t^{1/\alpha}$), then $\Gamma(t, z) \leq 2^{2n+3+[\alpha]}c_1 \equiv c_2$.

Let $|z| \geq 1$ ($|x| \geq t^{1/\alpha}$). Then, taking into account (2.14), we obtain

$$\Gamma(t, z) = \sum_{k=0}^p C_p^k |z|^k |G_0(t, z)| \leq \sum_{k=0}^p C_p^k |z|^k c |z|^{-p} \leq c_0, \quad c_0 = c \sum_{k=0}^p C_p^k, \quad p = 2n + 3 + [\alpha].$$

Thus,

$$|G_0(t, z)| \leq \tilde{c}(1 + |z|)^{-(2n+3+[\alpha])}, \quad \forall t > 1, \quad z \in \mathbb{R}, \quad z = t^{-1/\alpha}x,$$

where $\tilde{c} = \max\{c_0, c_2\}$, which is what we needed to establish.

Let $m \in \mathbb{N}$. Using the formula

$$G(t, x) = t^{-(2\nu+2)/\alpha} G_0(t, z), \quad z = t^{-1/\alpha}x,$$

we obtain

$$D_x^m G(t, x) = c_\nu t^{-(2\nu+2+m)/\alpha} D_z^m G_0(t, z) = t^{-(2\nu+2+m)/\alpha} (\Lambda_{1,m}(t, z) + \Lambda_{2,m}(t, z)),$$

where

$$\begin{aligned} \Lambda_{1,m}(t, z) &= c_n \sum_{k=0}^n b_k D_z^m (z^{-(n+k+1)} J_{1,k}(t, z)) \\ &= c_n \sum_{k=0}^n b_k \int_0^\infty Q(t, t^{-1/\alpha}\xi) \xi^{n-k+1} D_z^m (z^{-(n+k+1)} \sin(z\xi - \frac{\pi n}{2})) d\xi, \\ \Lambda_{2,m}(t, z) &= c_n \sum_{k=0}^n d_k D_z^m (z^{-(n+k+1)} J_{2,k}(t, z)) \\ &= c_n \sum_{k=0}^n d_k \int_0^\infty Q(t, t^{-1/\alpha}\xi) \xi^{n-k+1} D_z^m (z^{-(n+k+1)} \cos(z\xi - \frac{\pi n}{2})) d\xi, \quad z > 0. \end{aligned}$$

We estimate $|\Lambda_{1,m}(t, z)|$. Using the Leibniz formula for differentiating the product of two functions, we obtain

$$\begin{aligned} D_z^m \left(z^{-(n+k+1)} \sin \left(z\xi - \frac{\pi n}{2} \right) \right) &= \sum_{j=0}^m C_m^j \alpha_j z^{-(n+k+1+j)} \xi^{m-j} \sin \left(z\xi - \frac{\pi n}{2} + (m-j) \frac{\pi}{2} \right), \\ \alpha_j &= (-1)^j (n+k+1) \cdots (n+k+1+j). \end{aligned}$$

Thus,

$$\begin{aligned} \Lambda_{1,m}(t, z) &= c_n \sum_{k=0}^n b_k \sum_{j=0}^m C_m^j \alpha_j z^{-(n+k+1+j)} \int_0^\infty Q(t, t^{-1/\alpha}\xi) \xi^{n-k+1+m-j} \sin \left(z\xi - \frac{\pi n}{2} + (m-j) \frac{\pi}{2} \right) d\xi \\ &= c_n \sum_{k=0}^n b_k \sum_{j=0}^m C_m^j \alpha_j z^{-(n+k+1+j)} \Gamma_{1,k,m,j}(t, z), \end{aligned}$$

where

$$\Gamma_{1,k,m,j}(t, z) = \int_0^\infty Q(t, t^{-1/\alpha}\xi) \xi^{n-k+1+m-j} \sin \left(z\xi - \frac{\pi n}{2} + (m-j) \frac{\pi}{2} \right) d\xi.$$

Reasoning similarly to what was done in the case $m = 0$ and integrating by parts $s = n - k + 2 + [\alpha] + m - j$ times, we find that

$$|\Gamma_{1,k,m,j}(t, z)| \leq \frac{1}{|z|^s} \int_0^\infty |D_\xi^s(Q(t, t^{-1/\alpha}\xi) \xi^{n-k+1+m-j})| d\xi$$

$$\begin{aligned}
&\leq \frac{1}{|z|^s} \left[\int_0^\infty |D_\xi^s Q(t, t^{-1/\alpha} \xi)| \xi^{n-k+1+m-j} d\xi + \dots \right. \\
&\quad + \frac{(n-k+1+m-j)!}{(n-k+1+m-j-l)!} C_s^{s-l} \int_0^\infty |D_\xi^{s-l} Q(t, t^{-1/\alpha} \xi)| \xi^{n-k+1+m-j-l} d\xi + \dots \\
&\quad \left. + (n-k+1+m-j)! \int_0^\infty |D_\xi^{s-(n-k+1+m-j)} Q(t, t^{-1/\alpha} \xi)| d\xi \right], \quad z \neq 0.
\end{aligned}$$

From the properties of the function $Q(t, t^{-1/\alpha} \xi)$ it follows that all integrals are convergent. As in the case $m = 0$, we obtain

$$|\Gamma_{1,k,m,j}(t, z)| \leq \beta_{k,m,j} |z|^{-s}, \quad z = t^{-1/\alpha} x \neq 0, \quad t > 1, \quad s = n - k + 2 + [\alpha] + m - j.$$

The constants $\beta_{k,m,j}$ do not depend on t . From this we already obtain the estimate

$$\begin{aligned}
|\Lambda_{1,m}(t, z)| &\leq c_n \sum_{k=0}^n |b_k| \sum_{j=0}^m C_m^j |\alpha_j| \beta_{k,m,j} |z|^{-(n+k+1+j)} |z|^{-(n-k+2+[\alpha]+m-j)} \\
&= \tilde{\alpha}_m |z|^{-(2n+3+[\alpha]+m)} = \tilde{\alpha}_m |z|^{-(2\nu+2+[\alpha]+m)}, \quad z = t^{-1/\alpha} x \neq 0, \quad t > 1.
\end{aligned}$$

We estimate $|\Lambda_{2,m}(t, z)|$ similarly. As a result, we arrive at the inequalities

$$|D_z^m G_0(t, z)| \leq c_m'' (1 + |z|)^{-(2\nu+2+[\alpha]+m)} = c_m'' (t^{1/\alpha} + |x|)^{-(2\nu+2+[\alpha]+m)} t^{(2\nu+2+[\alpha]+m)/\alpha}, \quad t > 1.$$

Then

$$\begin{aligned}
|D_x^m G(t, x)| &\leq \tilde{c}_m t^{-(2\nu+2+m)/\alpha} t^{(2\nu+2+[\alpha]+m)/\alpha} (t^{1/\alpha} + |x|)^{-(2\nu+2+[\alpha]+m)} \\
&= \tilde{c}_m t^{[\alpha]/\alpha} (t^{1/\alpha} + |x|)^{-(2\nu+2+[\alpha]+m)}, \quad m \in \mathbb{Z}_+, \quad x \in \mathbb{R}, \quad t > 1,
\end{aligned}$$

where the constant $\tilde{c}_m > 0$ does not depend on t , which is what we needed to establish.

The case $0 < t \leq 1$ is considered similarly (using estimate (2.10)). As a result, we arrive at the estimate

$$|D_x^m G(t, x)| \leq \tilde{c}_m t^{-((\alpha-1)(m+\nu+3/2)+\alpha[\alpha]+\nu+1/2)/\alpha} (t^{1/\alpha} + |x|)^{-(2\nu+2+[\alpha]+m)}, \quad x \in \mathbb{R}, \quad 0 < t \leq 1,$$

where the constant $\tilde{c}_m > 0$ does not depend on t . Combining these inequalities into one, we obtain (2.11). \square

Remark 2.3. From the properties of the function $Q(t, \sigma)$ it follows that the function $G(t, x)$ is continuously differentiable as a function of the argument $t \in (0, +\infty)$. Moreover, since

$$Q(t, \sigma) = F_{B_\nu}[G(t, x)] = \int_0^\infty G(t, x) j_\nu(\sigma x) x^{2\nu+1} dx, \quad j_\nu(0) = 1,$$

we have

$$\int_0^\infty G(t, x) x^{2\nu+1} dx = Q(t, 0) = \left(\mu - \sum_{k=1}^m \mu_k \right)^{-1}.$$

Lemma 2.3. *The function $G(t, \cdot)$, $t \in (0, +\infty)$, as an abstract function of the parameter t with values in the space Ψ_α , is differentiable with respect to t .*

Proof. From the property of continuity of the Bessel transform it follows that to prove the statement it suffices to establish that the function $F_{B\nu}[G(t, \cdot)] = Q(t, \cdot)$, as an abstract function of the parameter t with values in the space Φ_α , is differentiable with respect to t . In other words, it is necessary to prove that the limiting relation

$$\Gamma_{\Delta t}(\sigma) := \frac{1}{\Delta t} [Q(t + \Delta t, \sigma) - Q(t, \sigma)] \longrightarrow \frac{\partial}{\partial t} Q(t, \sigma), \quad \Delta t \rightarrow 0,$$

holds in the sense of convergence with respect to the topology of the space Φ_α . Note that

$$\begin{aligned} \Gamma_{\Delta t}(\sigma) &= -|\sigma|^\alpha Q(t + \theta \Delta t, \sigma), \quad 0 < \theta < 1, \\ \Gamma_{\Delta t}(\sigma) - \frac{\partial}{\partial t} Q(t, \sigma) &= |\sigma|^{2\alpha} Q(t + \theta_1 \Delta t, \sigma) \theta \Delta t, \quad 0 < \theta_1 < 1. \end{aligned}$$

Taking into account the properties of the function $Q(t, \sigma)$, we prove that

$$\left\| \Gamma_{\Delta t} - \frac{\partial}{\partial t} Q \right\|_p \longrightarrow 0, \quad \Delta t \rightarrow 0, \quad \forall p \in \mathbb{Z}_+. \quad \square$$

Corollary 2.1. *The following formula is valid:*

$$\frac{\partial}{\partial t} (f * G(t, \cdot)) = f * \frac{\partial G(t, \cdot)}{\partial t}, \quad \forall f \in \Phi'_\alpha, \quad t > 0.$$

Proof. According to the definition of convolution of a generalized function with a test function

$$f * G(t, x) = \langle f_\xi, T_x^\xi G(t, x) \rangle = \langle f_\xi, T_x^\xi G(t, \xi) \rangle,$$

we have

$$\begin{aligned} \frac{\partial}{\partial t} (f * G(t, x)) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [(f * G(t + \Delta t, x)) - (f * G(t, x))] \\ &= \lim_{\Delta t \rightarrow 0} \left\langle f_\xi, \frac{1}{\Delta t} [T_x^\xi G(t + \Delta t, x) - T_x^\xi G(t, x)] \right\rangle. \end{aligned}$$

Due to Lemma 2.3, the limiting relation

$$\frac{1}{\Delta t} [T_x^\xi G(t + \Delta t, x) - T_x^\xi G(t, x)] \xrightarrow{\Delta t \rightarrow 0} \frac{\partial}{\partial t} T_x^\xi G(t, x)$$

holds in the sense of convergence with respect to the topology of the space Ψ_α , therefore, taking into account the continuity of the functional f ,

$$\begin{aligned} \frac{\partial}{\partial t} (f * G(t, x)) &= \left\langle f_\xi, \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [T_x^\xi G(t + \Delta t, x) - T_x^\xi G(t, x)] \right\rangle \\ &= \left\langle f_\xi, \frac{\partial}{\partial t} T_x^\xi G(t, x) \right\rangle = \left\langle f_\xi, T_x^\xi \frac{\partial}{\partial t} G(t, x) \right\rangle = f * \frac{\partial}{\partial t} G(t, x), \end{aligned}$$

which is what needed to be proved. \square

Lemma 2.4. *In the space Ψ'_α , the following relations hold:*

- 1) $G(t, \cdot) \rightarrow F_{B\nu}^{-1}[Q_2], \quad t \rightarrow +0;$
- 2)

$$\mu G(t, \cdot) - \sum_{k=1}^m \mu_k G(t_k, \cdot) \rightarrow \delta, \quad t \rightarrow +0. \quad (2.16)$$

Proof.

1) To prove the statement, it suffices to establish that $F_{B_\nu}[G(t, \cdot)] = Q_1(t, \cdot)Q_2(\cdot) \rightarrow Q_2(\cdot)$, $t \rightarrow +0$, in the space Φ'_α [1]. Toward this end, we take an arbitrary function $\varphi \in \Phi_\alpha$ and, using the fact that Q_2 is a multiplier in the space Φ_α [4], as well as the Lebesgue dominated convergence theorem, we find that

$$\begin{aligned} \langle Q_1(t, \cdot)Q_2(\cdot), \varphi \rangle &= \langle Q_1(t, \cdot), Q_2(\cdot)\varphi(\cdot) \rangle \\ &= \int_0^\infty Q_1(t, \sigma)Q_2(\sigma)\varphi(\sigma)\sigma^{2\nu+1} d\sigma \xrightarrow{t \rightarrow +0} \int_0^\infty Q_2(\sigma)\varphi(\sigma)\sigma^{2\nu+1} d\sigma = \langle 1, Q_2(\cdot)\varphi(\cdot) \rangle = \langle Q_2, \varphi \rangle \end{aligned}$$

(here, we understand $Q_1(t, \cdot)Q_2(\cdot)$ as a regular generalized function from the space Φ'_α). From this, statement 1) of Lemma 2.4 follows.

2) Taking into account statement 1) of Lemma 2.4, we obtain

$$\begin{aligned} \mu G(t, \cdot) - \sum_{k=1}^m \mu_k G(t_k, \cdot) &\xrightarrow{t \rightarrow +0} \mu F_{B_\nu}^{-1}[Q_2] - \sum_{k=1}^m \mu_k F_{B_\nu}^{-1}[Q_1(t_k, \cdot)Q_2(\cdot)] \\ &= F_{B_\nu}^{-1} \left[\mu Q_2(\cdot) - \sum_{k=1}^m \mu_k Q_1(t_k, \cdot)Q_2(\cdot) \right] = F_{B_\nu}^{-1} \left[\left(\mu - \sum_{k=1}^m \mu_k Q_1(t_k, \cdot) \right) Q_2(\cdot) \right] \\ &= F_{B_\nu}^{-1} \left[\left(\mu - \sum_{k=1}^m \mu_k Q_1(t_k, \cdot) \right) \left(\mu - \sum_{k=1}^m \mu_k Q_1(t_k, \cdot) \right)^{-1} \right] = F_{B_\nu}^{-1}[1] = \delta. \end{aligned}$$

Thus, relation (2.16) holds. \square

Remark 2.4. If $\mu = 1$, $\mu_1 = \dots = \mu_m = 0$, then problem (2.1), (2.2) reduces to the Cauchy problem for equation (2.1). In this case, $Q_2(\sigma) = 1$, $\forall \sigma \in \mathbb{R}$, $G(t, x) = F_{B_\nu}^{-1}[e^{-t|\sigma|^\alpha}]$ and $G(t, \cdot) \rightarrow F_{B_\nu}^{-1}[1] = \delta$, as $t \rightarrow +0$, in the space Ψ'_α .

Corollary 2.2. *Let $\omega(t, x) = f * G(t, x)$, $f \in \Psi'_{\alpha,*}$, $(t, x) \in \Omega$ (here, $\Psi'_{\alpha,*}$ is the class of convolutors in the space Ψ_α [4]). Then, in the space Ψ'_α , the following limiting relation holds:*

$$\mu \omega(t, \cdot) - \sum_{k=1}^m \mu_k \omega(t_k, \cdot) \longrightarrow f, \quad t \rightarrow +0. \quad (2.17)$$

Proof. We prove that the limiting relation

$$F_{B_\nu} \left[\mu \omega(t, \cdot) - \sum_{k=1}^m \mu_k \omega(t_k, \cdot) \right] \longrightarrow F_{B_\nu}[f], \quad t \rightarrow +0, \quad (2.18)$$

holds in the space Φ'_α . Since $f \in \Psi'_{\alpha,*}$, $G(t, \cdot) \in \Psi_\alpha$ for each $t > 0$, we have

$$F_{B_\nu}[\omega(t, \cdot)] = F_{B_\nu}[f * G(t, \cdot)] = F_{B_\nu}[f] \cdot F_{B_\nu}[G(t, \cdot)] = F_{B_\nu}[f] \cdot Q(t, \cdot).$$

Therefore, it is necessary to prove that

$$F_{B_\nu}[f] \left(\mu Q(t, \cdot) - \sum_{k=1}^m \mu_k Q(t_k, \cdot) \right) \longrightarrow F_{B_\nu}[f] \quad \text{as } t \rightarrow +0,$$

in the space Φ'_α . Since $Q_1(t, \cdot)Q_2(\cdot) \rightarrow Q_2(\cdot)$ as $t \rightarrow +0$, in the space Φ'_α (see the proof of statement 1) of Lemma 2.4), we have

$$\mu Q(t, \cdot) - \sum_{k=1}^m \mu_k Q(t_k, \cdot) \xrightarrow{t \rightarrow +0} \mu Q_2(\cdot) - \sum_{k=1}^m \mu_k Q_1(t_k, \cdot)Q_2(\cdot) = \left(\mu - \sum_{k=1}^m \mu_k Q_1(t_k, \cdot) \right) Q_2(\cdot) = 1$$

in the space Φ'_α . Thus, relation (2.18), and hence (2.17) hold in the corresponding spaces. The statement is proved. \square

Remark 2.5. The function $G(t, x)$, $(t, x) \in \Omega$, is a solution of equation (2.1). Indeed,

$$\frac{\partial}{\partial t} G(t, x) = -F_{B_\nu}^{-1} [|\sigma|^\alpha Q(t, \sigma)], \quad AG(t, x) = F_{B_\nu}^{-1} [|\sigma|^\alpha F_{B_\nu} [F_{B_\nu}^{-1} Q(t, \cdot)]] = F_{B_\nu}^{-1} [|\sigma|^\alpha Q(t, \sigma)],$$

whence we obtain

$$\frac{\partial}{\partial t} G(t, x) + AG(t, x) = 0, \quad (t, x) \in \Omega,$$

which is what needed to be established.

Henceforth, we call the function $G(t, x)$, $(t, x) \in \Omega$, the **fundamental solution** of the multipoint in time problem for equation (2.1).

From Corollary 2.2 it follows that the nonlocal multipoint in time problem for equation (2.1) can be posed as follows: find a function $u(t, x)$, $(t, x) \in \Omega$, that satisfies equation (2.1) and the condition

$$\mu \lim_{t \rightarrow +0} u(t, \cdot) - \sum_{k=1}^m \mu_k u(t_k, \cdot) = f, \quad f \in \Psi'_{\alpha, *}, \quad (2.19)$$

(the limiting relation (2.19) is considered in the space Ψ'_α , the restrictions on the parameters μ , μ_1, \dots, μ_m , t_1, \dots, t_m are the same as in the case of problem (2.1), (2.2)).

Theorem 2.1. *The nonlocal multipoint in time problem (2.1), (2.19) is well-posed, and the solution is given by the formula*

$$u(t, x) = f * G(t, x), \quad (t, x) \in \Omega,$$

where $u(t, \cdot) \in \Psi_\alpha$ for each $t > 0$.

Proof. Let us verify that the function $u(t, x)$, $(t, x) \in \Omega$, satisfies equation (2.1). Indeed (see Corollary 2.1),

$$\begin{aligned} \frac{\partial u(t, x)}{\partial t} &= \frac{\partial}{\partial t} (f * G(t, x)) = f * \frac{\partial G(t, x)}{\partial t}, \\ Au(t, x) &= F_{B_\nu}^{-1} [|\sigma|^\alpha F_{B_\nu} [f * G(t, x)]]. \end{aligned}$$

Since f is a convolutor in the space Ψ_α , we have

$$F_{B_\nu} [f * G(t, \cdot)] = F_{B_\nu} [f] F_{B_\nu} [G(t, \cdot)] = F_{B_\nu} [f] Q(t, \cdot).$$

Therefore,

$$\begin{aligned} Au(t, x) &= F_{B_\nu}^{-1} [|\sigma|^\alpha Q(t, \sigma) F_{B_\nu} [f]] = -F_{B_\nu}^{-1} \left[\frac{\partial}{\partial t} Q(t, \cdot) F_{B_\nu} [f] \right] \\ &= -F_{B_\nu}^{-1} \left[F_{B_\nu} \left[\frac{\partial}{\partial t} G(t, \cdot) \right] F_{B_\nu} [f] \right] = -F_{B_\nu}^{-1} \left[F_{B_\nu} \left[f * \frac{\partial G(t, \cdot)}{\partial t} \right] \right] = -f * \frac{\partial G(t, \cdot)}{\partial t}. \end{aligned}$$

From this it follows that the function $u(t, x)$, $(t, x) \in \Omega$, satisfies equation (2.1).

From Corollary 2.2 it follows that u satisfies condition (2.19) in the indicated sense. We also note that u depends continuously on the function $f \in \Psi'_{\alpha, *}$, since the convolution operation has the continuity property.

It remains to verify that problem (2.1), (2.19) has a unique solution. To do this, we consider the Cauchy problem

$$\frac{\partial v(t, x)}{\partial t} = A^* v(t, x), \quad (t, x) \in [0, t_0) \times \mathbb{R} \equiv \Omega', \quad 0 \leq t < t_0 < +\infty, \quad (2.20)$$

$$v(t, \cdot)|_{t=t_0} = \psi, \quad \psi \in \Psi'_{\alpha, *}, \quad (2.21)$$

where A^* is the restriction of the adjoint operator of the operator A to the space Ψ_α . Condition (2.21) is understood in the weak sense. The Cauchy problem (2.20), (2.21) is well-posed, and the solution is given by the formula

$$v(t, x) = \psi * G^*(t, x), \quad G^*(t, x) = F_{B_\nu}^{-1} [\exp\{(t - t_0)|\sigma|^\alpha\}],$$

where $v(t, \cdot) \in \Psi_\alpha$ for each $t \in [0, t_0)$.

Let $Q_{t_0}^t : \Psi'_{\alpha,*} \rightarrow \Psi_\alpha$ be the operator that associates to the functional $\psi \in \Psi'_{\alpha,*}$ the solution of problem (2.20), (2.21). The operator $Q_{t_0}^t$ is linear and continuous, defined for arbitrary t and t_0 such that $0 \leq t < t_0 < +\infty$ and has the properties

$$\forall \psi \in \Psi'_{\alpha,*} : \frac{dQ_{t_0}^t \psi}{dt} = A^* Q_{t_0}^t \psi, \quad \lim_{t \rightarrow t_0} Q_{t_0}^t \psi = \psi$$

(the limit is considered in the space Ψ'_α).

Consider the solution $u(t, x)$, $(t, x) \in \Omega$, of problem (2.1), (2.19), which we understand as a regular functional from the space $\Psi'_{\alpha,*} \supset \Psi_\alpha$. We prove that problem (2.1), (2.19) may have only a unique solution in the space $\Psi'_{\alpha,*}$. To do this, it suffices to prove that the only solution of equation (2.1) with zero initial condition can be the functional $u(t, x) = 0$ (for each $t \in (0, +\infty)$). We apply the functional u to the function $Q_{t_0}^t \psi \in \Psi_\alpha$, where ψ is an arbitrarily fixed element from the space $\Psi_\alpha \subset \Psi'_{\alpha,*}$. Differentiating with respect to t and using equations (2.1), (2.20), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \langle u(t, \cdot), Q_{t_0}^t \psi \rangle &= \left\langle \frac{\partial u}{\partial t}, Q_{t_0}^t \psi \right\rangle + \left\langle u, \frac{\partial Q_{t_0}^t \psi}{\partial t} \right\rangle \\ &= \langle -Au, Q_{t_0}^t \psi \rangle + \langle u, A^* Q_{t_0}^t \psi \rangle = -\langle Au, Q_{t_0}^t \psi \rangle + \langle Au, Q_{t_0}^t \psi \rangle = 0, \quad t \in [0, t_0). \end{aligned}$$

Therefore, $\langle u(t, \cdot), Q_{t_0}^t \psi \rangle$ is a constant. From the properties of abstract functions it follows that

$$\lim_{t \rightarrow t_0} \langle u(t, \cdot), Q_{t_0}^t \psi \rangle = \langle u(t_0, \cdot), \psi \rangle = \text{const} \equiv c, \quad c = c(t_0),$$

at an arbitrary point $t_0 \in (0, +\infty)$. Therefore, if in (2.19) $f = 0$, then

$$\mu \lim_{t \rightarrow +0} \langle u(t, \cdot), \psi \rangle - \sum_{k=1}^m \mu_k \langle u(t_k, \cdot), \psi \rangle = \mu c_0 - \sum_{k=1}^m \mu_k c_k = 0.$$

From this it follows that $c_0 = c_1 = \dots = c_m = 0$. Indeed, suppose this is not the case. For example, $c_0 \neq 0$. Then we have the relation $\mu - \sum_{k=1}^m \mu_k \alpha_k = 0$, where $\alpha_k = c_k/c_0$, that is, $\mu = \sum_{k=1}^m \mu_k \alpha_k$. Since α_k are arbitrary constants, while by assumption μ, μ_1, \dots, μ_m are fixed parameters, and $\mu > \sum_{k=1}^m \mu_k$, the resulting contradiction proves that $c_0 = 0$. Similarly, we prove that $c_1 = \dots = c_m = 0$. Thus, $\langle u(t_0, \cdot), \psi \rangle = 0$ for arbitrary $\psi \in \Psi_\alpha$, that is, $u(t_0, x)$ is the zero functional from the space $\Psi'_{\alpha,*}$. Since $t_0 \in (0, +\infty)$ and t_0 is chosen arbitrarily, we have $u(t, x) = 0$ for all $t \in (0, +\infty)$. \square

Theorem 2.2. *Let $u(t, x)$, $(t, x) \in \Omega$, be the solution of problem (2.1), (2.19) with initial function $f \in \Psi'_{\alpha,*}$ having compact support (that is, $\text{supp } f$ is a bounded set in \mathbb{R}). Then $u(t, x) \rightarrow 0$ as $t \rightarrow +\infty$ uniformly on \mathbb{R} .*

Proof. Let $\text{supp } f \subset [-a, a] \subset [-b, b] \subset \mathbb{R}$. Consider an even function $\varphi \in \Psi_\alpha$ such that $\varphi(x) = 1$ for $x \in [-a, a]$, and $\text{supp } \varphi \subset [-b, b]$. Such a function exists, since the space Ψ_α contains even functions with compact support. The function $G(t, x)$, $(t, x) \in \Omega$ — the fundamental solution of problem (2.1), (2.19) — is an even function of the variable x for each $t > 0$. We represent the function $u(t, x)$ in the form

$$u(t, x) = \langle f_\xi, \varphi(\xi) G(t, x - \xi) \rangle + \langle f_\xi, \gamma(\xi) G(t, x - \xi) \rangle,$$

where $\gamma = 1 - \varphi$. Since $\text{supp } (\gamma(\xi) G(t, x - \xi)) \cap \text{supp } f = \emptyset$, we have

$$u(t, x) = t^{-(2\nu+2)/\alpha} \langle f_\xi, t^{(2\nu+2)/\alpha} \varphi(\xi) G(t, x - \xi) \rangle.$$

The generalized function $f \in \Psi'_{\alpha,*} \subset \Psi'_\alpha = \bigcup_{p=0}^{\infty} \Psi'_{p,\alpha}$ has finite order $p \in \mathbb{Z}_+$, therefore,

$$|u(t, x)| \leq t^{-(2\nu+2)/\alpha} \|f\|_p \cdot \|\Gamma_{t,x}\|_p,$$

where $\Gamma_{t,x}(\xi) = t^{(2\nu+2)/\alpha} \varphi(\xi) G(t, x - \xi)$. Note that $\Gamma_{t,x}(\xi) = 0$ for $\xi \in \mathbb{R} \setminus [-b, b]$. To prove the stated claim, it suffices to establish that $\Gamma_{t,x}(\xi)$ is bounded in the norm of the space $\Psi_{p,\alpha}$, that is, that $\|\Gamma_{t,x}\|_p \leq c_p$, where the constant $c_p > 0$ does not depend on t and x ($t > 1$, $x \in \mathbb{R}$). For this, we use the estimate

$$\begin{aligned} |D_\xi^l G(t, x - \xi)| &\leq c_l t^{[\alpha]/\alpha} (t^{1/\alpha} + |x - \xi|)^{-(2\nu+2+[\alpha]+l)} \\ &\leq c_l t^{[\alpha]/\alpha} t^{-(2\nu+2+[\alpha]+l)/\alpha} \leq c_l t^{-(2\nu+2+l)/\alpha} \leq c_l t^{-(2\nu+2)/\alpha}, \quad l \in \mathbb{Z}_+, \end{aligned} \quad (2.22)$$

which holds for $t > 1$, $x \in \mathbb{R}$, $\xi \in [-b, b]$ and follows from (2.11). Since $\Gamma_{t,x}(\xi) = 0$ for $\xi \in \mathbb{R} \setminus [-b, b]$, taking into account (2.22), we obtain

$$\begin{aligned} \|\Gamma_{t,x}\|_p &= t^{(2\nu+2)/\alpha} \sup_{\xi \in [-b, b]} \left\{ \sum_{k=0}^p (1 + |\xi|)^{2\nu+2+[\alpha]+k} |(\varphi(\xi) G(t, x - \xi))^{(k)}| \right\} \\ &\leq t^{(2\nu+2)/\alpha} \left\{ \sum_{k=0}^p (1 + b)^{2\nu+2+[\alpha]+k} \sum_{l=0}^k C_k^l |\varphi^{(l)}(\xi)| |G^{(k-l)}(t, x - \xi)| \right\} \leq c_p \end{aligned}$$

(here, we used that $|\varphi^{(l)}(\xi)| \leq c_l^1$, $l \in \{0, 1, \dots, k\}$, $\xi \in [-b, b]$). Therefore,

$$|u(t, x)| \leq \tilde{c}_p t^{-(2\nu+2)/\alpha}, \quad t > 1, \quad x \in \mathbb{R},$$

where $\tilde{c}_p = c_p \|f\|_p$, from which it follows that $u(t, x) \rightarrow 0$ as $t \rightarrow +\infty$ uniformly on \mathbb{R} , which is what needed to be proved. Other possible cases of the location of $\text{supp } f$ on \mathbb{R} are studied similarly. \square

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