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NORMALIZING FACTOR ROLE IN NUMERICAL SOLUTION OF THE SEISMIC WAVES PROPAGATION PROBLEM

Abstract. In this paper, we consider the problem of seismic wave propagation based on the homogeneous Helmholtz equation with two cylindrical tunnel inclusions under perfectly welded interface conditions. The exact analytical solution of the model is obtained using a boundary method that involves series expansions of incident and reflected SH waves in terms of cylindrical wave functions, as well as coordinate transformations between different reference systems. We also examine the convergence of the approximate solution of the truncated system to the solution of the corresponding infinite system, and compute the relative error. This brief communication focuses on the role of introducing a normalizing coefficient in the truncation order of the resulting infinite block matrix equation in the study of seismic wave propagation boundary value problems for cylindrical inclusions in a half-space.

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Key words and phrases. Scattering of SH waves, Helmholtz equation, Graf's Theorem, two underground structures, cylindrical inclusions.

რეზიუმე. ნაშრომში განხილულია სეისმური ტალღის გავრცელების ამოცანა, რომელიც დაფუძნებულია ერთგვაროვან ჰელმჰოლცის განტოლებაზე გვირაბის ტიპის ორი ცილინდრული ჩანართით იდეალური გადაბმის პირობებით. მოდელის ზუსტი ანალიტიკური ამოხსნა მიღებულია სასაზღვრო მეთოდის გამოყენებით, რომელიც გულისხმობს დაცემული და არეკლილი SH ტალღების გაშლას ცილინდრული ტალღური ფუნქციების მწკრივებად, ასევე კოორდინატების გარდაქმნებს ათვლის სხვადასხვა სისტემებს შორის. ასევე გამოკვლეულია წაკვეთილი სისტემის მიახლოებითი ამონახსნის კრებადობა შესაბამისი უსასრულო სისტემის ამონახსნისკენ და გამოთვლილია ფარდობითი ცდომილება. ნაშრომი ძირითადად ფოკუსირებულია ქცევა ცილინდრულ-ჩანართებიან ნახევარსივრცეში სეისმური ტალღის გავრცელების სასაზღვრო ამოცანების შესწავლისას წარმოქმნილი უსასრულო ბლოკური მატრიცული განტოლების წაკვეთის რიგის განსაზღვრისას ნორმალიზაციის კოეფიციენტის შემოღების როლზე.

1 Introduction

Every day, there are about fifty earthquakes worldwide that are strong enough to be felt locally, and every few days an earthquake occurs that is capable of damaging structures. Each event radiates seismic waves that travel throughout the Earth, and several earthquakes per day produce distant ground motions that, although too weak to be felt, are readily detected with modern instruments anywhere on the globe [23]. Of course, it is extremely important to understand the physical processes that cause earthquakes and to seek ways to reduce their destructive impacts on humanity.

Early thinking about earthquakes was, as one might expect, superstitious and not very scientific. Seismology is a comparatively young science that has been studied quantitatively for only about a century. Yet the information it provides today has widely varying degrees of uncertainty. Some parameters, such as the average compressional-wave travel time through the mantle, are known to within a fraction of a percent, while others, such as the degree of seismic energy damping within the inner core, are still known only approximately. The average radial seismic velocity structure of the Earth has been understood fairly well for more than fifty years, and the locations and seismic radiation patterns of earthquakes are now routinely mapped, but many important aspects of earthquake physics remain a mystery [23].

Robert Mallet, an Irish engineer who traveled to Italy to study the destruction caused by the large earthquake near Naples in 1857, made the first significant attempt at observational seismology. His work articulated the idea that earthquakes radiate seismic waves outward from a focus point (now called the hypocenter), and that they can be located by projecting these waves backward to their source [18]. His concept was sound, as were his suggestions that observatories be established to monitor earthquakes and his experiments on measuring seismic velocities using artificial sources [19].

However, modern problems associated with realistic sources and various complex structures motivate complex mathematical treatments, improvement, refinement of old models, the search for new ones, and the extensive use of powerful computers.

Tunnel facilities play an important role in social development and seismic engineering. As a result, the seismic design and analysis of tunnels [10], structural health diagnostics [11], and excavation construction [6, 9] are important and active research topics. A detailed review of the relevant literature is provided in [30], where it is also highlighted the fact that many researchers have studied the dynamic response of topography and underground structures to seismic waves [3, 5, 13–16, 31]. Among the analytical approaches, the most widely used methods include the wave function expansion method [21], the Graf addition theorem and region-matching method [27–29], integral equation and integral transformation methods, perturbation methods, complex variable function methods [17], Green's function methods, and separation of variables.

In this paper, we consider the problem of seismic wave propagation based on the homogeneous Helmholtz equation with two tunnel inclusions under perfectly welded interface conditions. The exact analytical solution of the model is obtained by a boundary method that involves a series expansion of incident and reflected SH ways in terms of cylindrical wave functions, as well as coordinate transformations between any two reference systems. The expression for the standing wave is derived using the separation-of-variables method, and the scattered wave field in the homogeneous half-space is obtained via the mirror-image method. The unknown coefficients in the wave field representation are then determined from the boundary conditions.

Since an important feature of the resulting infinite system for analysis is the fact that the matrix elements decrease in columns but grow exponentially in rows, then when combining these oppositely behaving coefficients, the overall matrix equation turns out to be unbalanced and therefore the question of choosing the truncation number N here is not as simple as it might seem at first glance. We found that to eliminate the defect discussed above, it is sufficient to introduce a normalizing coefficient of a certain type. Within the framework of this article we demonstrate the behavior of the error function on the truncation order of infinite block-type matrix equation in two situations: without a normalizing factor, and with our idea of introducing such a normalizing coefficient. As a result, the simple and clear argument we propose here for the importance of using a normalizing factor is as follows: the error of a solution without a weighting factor stops decreasing and starts increasing, whereas with a weighting factor the error reaches the machine precision of the calculations.

2 Statement of the problem

2.1 Equation

We investigate the boundary value problem for a wide class of elliptic partial differential equations that is used in the study of stationary oscillating processes:

$$\Delta H_j^{(q)} + k_j^2 H_j^{(q)} = 0 \tag{2.1}$$

where $q = 1, 2,\ j = s, c,\ H_s$ and H_c are the displacements in the half-space and inside the cavity, respectively; k_j is the piecewise wavenumber, $k_s = \frac{\omega}{\beta_s}$ is the wavenumber associated with shear waves; $k_c = \frac{\omega}{\beta_c}$, respectively; $\beta_s = \sqrt{\frac{\mu_s}{\rho_s}}$ is the complex shear wave velocity of the half-space with mass density, shear modulus and quality factor given by μ_c, ρ_c.

As is well known, equation (2.1) is named after Hermann von Helmholtz, and any problems related to steady-state oscillations (mechanical, acoustical, thermal, electromagnetic) lead to the two-dimensional Helmholtz equation.

2.2 Conditions

We assume that two underground structures are identical infinite circular cylinders, each of radius a_q, $q = 1, 2$, and that these inclusions completely embedded in a half-space filled with a linearly elastic homogeneous medium that models the behavior of an idealized soil (isotropic and homogeneous). It is convenient to introduce local Cartesian (x_q, y_q) and polar coordinates (r_q, ϕ_q), $q = 1, 2$, with the origins at the centers of each of the two cylinders, and also the global coordinates with origin, without loss of generality, at the center of the first of them.

It is well-known that in applied mathematics and theoretical physics, the radiation condition, introduced by Arnold Sommerfeld [25], is used to select a particular solution to the Helmholtz equation and is closely related to the limiting absorption principle (1905) and the limiting amplitude principle (1948). The boundary condition established by this principle essentially selects a solution of some wave equations that radiates only outwards from the known sources. Instead of allowing arbitrary inbound waves from the infinity propagating in instead detracts from them. distracts Instead of allowing arbitrary incoming waves to propagate from infinity, it distract from them. Arnold Sommerfeld defined in [26] the condition of radiation for a scalar field satisfying the Helmholtz equation as *"the sources must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into ... the field"*. Accordingly, to our homogeneous Helmholtz equation (2.1) we add the Sommerfeld radiation condition at infinity [26]:

$$\Big(\frac{\partial}{\partial r} + ik_j\Big) H_j^{(q)} = o(\sqrt{r}), \quad r \to \infty, \quad q = 1, 2, \quad j = s, c. \tag{2.2}$$

So, let us once again emphasize the fact that the Sommerfeld condition is applied at infinity and, when added to the statement of the boundary value problem, singles out only the solution which represents "outgoing" (rather than "incoming" or "standing") waves in the physical applications.

Now, let's proceed to the next important condition for our problem statement, the condition of local power finiteness for an arbitrary bounded domain $\Omega \subset \mathbb{R}^2$:

$$\int\limits_{\Omega} \Big[|\nabla H_j^{(q)}|^2 + k_j^2 |H_j^{(q)}|^2 \Big] ds < \infty, \quad q = 1, 2, \quad j = s, c. \tag{2.3}$$

As is known, conditions (2.2), (2.3) guarantee the uniqueness of the solution of the boundary-value problem for all real-valued k_j.

Finally, it is time to describe the total displacement in domain we need to match H_s and H_c on the boundary of the cavity, namely,

$$H_c^{(q)}\big|_{r_q = a-0} = H_s^{(q)}\big|_{r_q = a+0} \tag{2.4}$$

and

$$\left.\frac{\partial H_c^{(q)}}{\partial r_q}\right|_{r_q=a-0} = \frac{\mu_s}{\mu_c}\left.\frac{\partial H_s^{(q)}}{\partial r_q}\right|_{r_q=a+0}. \tag{2.5}$$

2.3 Graf's Theorem

In this work, we will rely on the technique of images and the use of a generalized version of Graf's Addition Theorem, easy to specialize to different kinds of applications of seismic wave propagation in elastic media. Provided the wave solution in terms of $H_m^{(2)}(k|x-y|)$ and seek the equivalent solution in terms of the polar coordinates $(|y|, \theta_y)$, applying the Graf's Theorem, as provided by Abramowitz and Stegun [1] and end up for the problem we are interested in, thanks to C. Smerzini, J. Aviles, R. Paolucci and F. J. Sanchez-Sesma in [24], we obtain a compact expression

$$H_m^{(2)}(k|x-y|)e^{in\phi} = \sum_{n=-\infty}^{+\infty} H_{m+n}^{(2)}\big(k\max\{|x|,|y|\}\big)e^{i(m+n)\theta_x} J_n\big(k\min\{|x|,|y|\}\big)e^{-in\theta_y},$$

where, as usual, standardly designated by $H_n^{(2)}(\,\cdot\,)$ is the Hankel's function of the second kind and order n and in conventional notations, $J_n(\,\cdot\,)$ is the Bessel's function of the first kind and order n.

3 The main result

The total field for the problem is

$$H^{tot} = \begin{cases} H^{int(j)}, & r_j < a_j, \\ H^0 + H^{ext}, & r_j > a_j, \end{cases}$$

and taking this fact into account, we rewrite our boundary conditions (2.4) and (2.5), which are transformed into the conditions of the following form:

$$H^{int(j)}(a_j,\phi_j) = H^0(a_j,\phi_j) + H^{ext}(a_j,\phi_j) \tag{3.1}$$

and

$$\left.\frac{\partial H^{int(j)}(r_j,\phi_j)}{\mu_s/\mu_c\,\partial r_j}\right|_{r_j=a_j} = \left.\frac{\partial H^0(r_j,\phi_j)}{\partial r_j}\right|_{r_j=a_j} + \left.\frac{\partial H^{ext}(r_j,\phi_j)}{\partial r_j}\right|_{r_j=a_j}, \tag{3.2}$$

respectively, the displacement wavefield reflacted and trapped within the inclusion represented as

$$H^{int(i)}(r,\phi) = \sum_{n=-\infty}^{+\infty} y_n^{(i)} J_n(k_c r_i)e^{in\phi_i}, \quad r_i < a_i, \quad i = 1,2, \tag{3.3}$$

where $y_n^{(i)}$ are unknown coefficients and, as before, $J_n(\,\cdot\,)$ is the Bessel's function of the first kind and order n.

The incident field in the polar coordinate system

$$H^0 = Ae^{ikr\cos(\phi-\beta)}$$

can be expressed using the Fourier transformation for the plane SH-waves with amplitude $A = 1$ as follows:

$$H^0(r,\phi) = \sum_{n=-\infty}^{+\infty} U_n^{0,j}(r_j)e^{in\phi_j}, \quad j = 1,2, \tag{3.4}$$

where

$$U_n^{0,j}(r_j) = i^n J_n(k_s r_j)e^{-in\beta}, \tag{3.5}$$

and ϕ is the incident angle.

The form of the diffracted fields induced by the inclusion is as follows:

$$H^{ext}(r,\phi) = \sum_{n=-\infty}^{+\infty} z_n^{(1)} H_n^{(2)}(k_s r_1) e^{in\phi_1} + \sum_{n=-\infty}^{+\infty} z_n^{(2)} H_n^{(2)}(k_s r_2) e^{in\phi_2}, \tag{3.6}$$

where $z_n^{(i)}$ are unknown coefficients; $r_i > a_i$, $i = 1, 2$, and, as usual, $H_n^{(2)}(\,\cdot\,)$ is the Hankel's function of the second kind and order n.

Next, applying a generalized form of Graf's addition theorem (see Subsection 2.3) to the diffracted wave field for the problem considered above, we obtain

$$H_n^{(2)}(k_s r_q) e^{in\phi_q} = \sum_{m=-\infty}^{+\infty} J_m(k_s r_j) H_{m-n}^{(2)}(k_s d_{jq}) e^{i(n-m)\psi_{jq}} e^{im\phi_j}, \quad r_q > r_j. \tag{3.7}$$

For the simplicity of notation, we define the following quantities: $H_n^{(2)} \equiv H_n$, $H_{n-m}^{(2)} \equiv H_{n-m}$, and taking into account our assumptions, equality (3.7) has the following presentation:

$$H_m(k_s r_j) e^{im\phi_j} = \sum_{n=-\infty}^{+\infty} H_{n-m}(k_s L) J_n(k_s r_p) e^{i(m-n)\psi_{pj}} e^{in\phi_p}, \quad r_p < L, \quad j \neq p = 1, 2. \tag{3.8}$$

Applying the boundary conditions (3.1), (3.2) and employing the series expansions (3.3)–(3.8), we have

$$\sum_{q=-\infty}^{+\infty} \Big[U_q^{0,1}(a_1) + z_q^{(1)} H_q(k_s a_1) + \sum_{n=-\infty}^{+\infty} z_n^{(2)} J_q(k_s a_1) H_{q-n}(k_s L_{1,2}) e^{i(n-q)\phi_{1,2}} \Big] e^{iq\phi_1} = \sum_{q=-\infty}^{+\infty} y_q^{(1)} J_q(k_c a_1) e^{iq\phi_1}$$

and

$$\sum_{q=-\infty}^{+\infty} \left[\frac{\partial U_q^{0,1}}{\partial r_1}\Big|_{r_1=a_1} + z_q^{(1)} k_s H_q'(k_s a_1) + \sum_{n=-\infty}^{+\infty} z_n^{(2)} k_s J_q'(k_s a_1) H_{q-n}(k_s L_{1,2}) e^{i(n-q)\phi_{1,2}} \right] e^{iq\phi_1} = \sum_{q=-\infty}^{+\infty} y_q^{(1)} k_c J_q'(k_c a_1) e^{iq\phi_1}.$$

Equating the corresponding Fourier coefficients, we obtain

$$U_q^{0,1}(a_1) + z_q^{(1)} H_q(k_s a_1) + \sum_{n=-\infty}^{+\infty} z_n^{(2)} J_q(k_s a_1) H_{q-n}(k_s L_{1,2}) e^{i(n-q)\phi_{1,2}} = y_q^{(1)} J_q(k_c a_1)$$

and

$$\frac{\partial U_q^{0,1}}{\partial r_1} + z_q^{(1)} k_s H_q'(k_s a_1) + \sum_{n=-\infty}^{+\infty} z_n^{(2)} k_s J_q'(k_s a_1) H_{q-n}(k_s L_{1,2}) e^{i(n-q)\phi_{1,2}} = y_q^{(1)} k_c J_q'(k_c a_1).$$

Obviously,

$$\Big[U_q^{0,1}(a_1) + z_q^{(1)} H_q(k_s a_1) + \sum_{n=-\infty}^{+\infty} z_n^{(2)} J_q(k_s a_1) H_{q-n}(k_s L_{1,2}) e^{i(n-q)\phi_{1,2}} \Big] k_c J_q'(k_c a_1) = \Big[\frac{\partial U_q^{0,1}}{\partial r_1} + z_q^{(1)} k_s H_q'(k_s a_1) + \sum_{n=-\infty}^{+\infty} z_n^{(2)} k_s J_q'(k_s a_1) H_{q-n}(k_s L_{1,2}) e^{i(n-q)\phi_{1,2}} \Big] J_q(k_c a_1).$$

After elementary algebraic transformations, we get

$$z_q^{(1)}\left(H_q(k_sa_1)k_cJ_q'(k_ca_1)-k_sH_q'(k_sa_1)J_q(k_ca_1)\right)$$
$$+\sum_{n=-\infty}^{+\infty} z_n^{(2)}\Big\{J_q(k_sa_1)k_cJ_q'(k_ca_1)-k_sJ_q'(k_sa_1)J_q(k_ca_1)\Big\}H_{q-n}(k_sL_{1,2})e^{i(n-q)\phi_{1,2}}$$
$$=\frac{\partial U_q^{0,1}}{\partial r_1}J_q(k_ca_1)-U_q^{0,1}(a_1)k_cJ_q'(k_ca_1),$$

and taking into account (3.5), we obtain

$$z_q^{(1)}+\sum_{n=-\infty}^{+\infty} z_n^{(2)}\frac{\big\{J_q(k_sa_1)k_cJ_q'(k_ca_1)-k_sJ_q'(k_sa_1)J_q(k_ca_1)\big\}H_{q-n}(k_sL_{1,2})e^{i(n-q)\phi_{1,2}}}{H_q(k_sa_1)k_cJ_q'(k_ca_1)-k_sH_q'(k_sa_1)J_q(k_ca_1)}$$
$$=\frac{i^q\big[J_q'(k_sa_1)J_q(k_ca_1)-k_cJ_q(k_sa_1)J_q'(k_ca_1)\big]e^{-iq\beta}}{H_q(k_sa_1)k_cJ_q'(k_ca_1)-k_sH_q'(k_sa_1)J_q(k_ca_1)},$$

or, equivalently, in the compact form

$$Z+PY=Q, \tag{3.9}$$

where, for $i=1,2$,

$$Z=\{Z^{(i)}\},\ \ Z^{(i)}=\{z_m^{(i)}\}_{m=-\infty}^{+\infty},\quad Y=\{Y^{(i)}\},\ \ Y^{(i)}=\{y_m^{(i)}\}_{m=-\infty}^{+\infty},$$
$$P=\{P^{(i)}\},\ \ P^{(i)}=\{P_m^{(i)}\}_{m=-\infty}^{+\infty},\quad Q=\{Q^{(i)}\},\ \ Q^{(i)}=\{Q_m^{(i)}\}_{m=-\infty}^{+\infty},$$
$$P_m^{(i)}=\frac{J_m(k_sa_i)k_cJ_m'(k_ca_i)-k_sJ_m'(k_sa_i)J_m(k_ca_i)}{H_m(k_sa_i)k_cJ_m'(k_ca_i)-k_sH_m'(k_sa_i)J_m(k_ca_i)}\,H_{m-n}(k_sL_{1,2})e^{i(n-m)\phi_{1,2}},$$
$$Q_m^{(i)}=\frac{i^m\big[J_m'(k_sa_i)J_m(k_ca_i)-k_cJ_m(k_sa_i)J_m'(k_ca_i)\big]}{H_m(k_sa_i)k_cJ_m'(k_ca_i)-k_sH_m'(k_sa_i)J_m(k_ca_i)}\,e^{-im\beta}.$$

Note that in the block-type matrix equation (3.9) each block is infinite. The widespread use of matrix equations like (3.9) in a significant number of previous studies, which have been employed to generate numerical results, appear reasonable. However, we should not forget the so-called phenomenon of "accumulation of round-off errors" when choosing the block truncation number N. It should be noted an important feature of the resulting system, namely, the fact that the matrix elements in (3.9) decay along the columns (i.e., with index n), but, exponentially grow along the lines (i.e., with index m).

In mathematical terms, convergence refers to the ability of a numerical method to minimize computational errors as the problem size (in this case, the matrix size) increases. As the problem grows, a convergent method should produce solutions that approach the exact values, with decreasing error, regardless of how large the system becomes. However, a more careful and accurate analysis shows that matrix equations of the form (3.9) cannot guarantee this type of convergence. It is important not to lose the key distinction between convergence and accuracy: while accuracy refers to how close a solution is to the true value, convergence involves the behavior of the error as the problem size grows. In this case, many numerical methods may appear to work correctly for small matrices or produce reasonable results for the first few digits of a solution, but they can fail when higher accuracy is required for larger matrices or more complex problems. So, the problem becomes more apparent when numerical schemes that seem to work well in the initial stages (providing the first few correct results) begin to diverge as the precision requirements increase. These divergent methods may exhibit significant errors when extended to more accurate calculations, which is why relying on them without further scrutiny can lead to incorrect or unreliable results in the long run. Therefore, it is crucial to distinguish between methods that merely appear accurate on the surface and those that guarantee convergence, ensuring that the errors do not grow uncontrollably as the problem scales up. This underscores the importance of carefully assessing the stability and convergence behavior, especially in the context of larger, more complex systems.

Thus, the main reason for which we stop to discuss this point is easily understood: the boundary conditions at the i-th contour couple two sets of coefficients in each azimuthal order n: $y_n^{(i)} J_n(k_c a)$ and $z_n^{(i)} H_n^{(2)}(k_s a)$. To satisfy these conditions for large n, the coefficients $y_n^{(i)}$ must balance the exponential decay of the Bessel functions for $n >> k_s a$, while the coefficients $z_n^{(i)}$ must balance the exponential growth of the Hankel functions. When these oppositely behaving coefficients are assembled into a single matrix equation, the resulting system becomes imbalanced. Consequently, the choice of truncation requires special attention and careful control.

To eliminate the defect discussed above, various attempts have been made to introduce a normalizing factor. For example, in [12] and [20], the Bessel function was used as a normalizing element:

$$x_m^{(i)} = \frac{z_n^{(i)}}{J_n(k_c a)}\,.$$

However, it is important to note that this approach introduces special points that coincide with the zeros of the Bessel function. So, our new idea is to use asymptotics that are sufficient to get the same result on convergence and at the same time, which is very important, do not give rise to the problem of the appearance of singular points. Thus, we have found that the defect of earlier publications can be fixed by introducing a normalizing coefficient of the following type:

$$x_n^{(1,2)} = z_n^{(1,2)} w_n, \tag{3.10}$$

where

$$w_{n<0} = (-1)^n w_{n>0}, \quad w_{n>0} = n!\Big(\frac{2}{k_s a}\Big)^n.$$

In this way, as a result, we obtain the matrix equation:

$$X + AX = B, \tag{3.11}$$

where, for $i = 1, 2$,

$$X = \{X^{(i)}\}, \quad X^{(i)} = \{x_m^{(i)}\}_{m=-\infty}^{+\infty},$$
$$A = \{A^{(i)}\}, \quad A^{(i)} = \{A_{n,m}^{(i)}\}_{n,m=-\infty}^{+\infty}, \quad B = \{B^{(i)}\}, \quad B^{(i)} = \{B_m^{(i)}\}_{m=-\infty}^{+\infty},$$
$$A_{n,m}^{(i)} = \frac{\big\{J_m(k_s a_i) k_c J'_m(k_c a_i) - k_s J'_m(k_s a_i) J_m(k_c a_i)\big\} H_{m-n}(k_s L_{1,2}) w_n}{\big\{H_m(k_s a_i) k_c J'_m(k_c a_i) - k_s H'_m(k_s a_i) J_m(k_c a_i)\big\} w_m}\, e^{i(n-m)\phi_{1,2}},$$
$$B_m^{(i)} = \frac{i^m\big[J'_m(k_s a_i) J_m(k_c a_i) - k_c J_m(k_s a_i) J'_m(k_c a_i)\big]}{\big\{H_m(k_s a_i) k_c J'_m(k_c a_i) - k_s H'_m(k_s a_i) J_m(k_c a_i)\big\} w_m} e^{-im\beta}.$$

Since well-known infinite systems such as (3.11) cannot, in general, be solved directly, truncation to a finite size is therefore necessary to obtain a solution at any point. In this approach, the order of the expansions is set equal to N, so that the number of unknown coefficients becomes $2N + 1$. We will check that the solutions are robust with respect to the choice of N. It is worth noting that in earlier works, authors typically stopped at rather small values of N, usually around 6–10, which were deemed sufficient to obtain convergent solutions.

To check the code convergence for a varying matrix-block truncation order N for our problem and clearly see the difference between truncation without a normalizing factor and with our idea of introducing such a normalizing coefficient and visualize its rate, we have computed the relative error:

$$err(N) = \frac{\|Z^N - Z^{N_{max}}\|}{\|Z^{N_{max}}\|}\,,$$

where the set should be understood as follows:

$$Z^N = \Big\{\underbrace{0, \ldots, z^1_{-N}, z^1_{-N+1}, \ldots, z^1_{N-1}, z^1_N, 0, \ldots, 0}_{2N_{max}+1}, \underbrace{0, \ldots, z^2_{-N}, z^2_{-N+1}, \ldots, z^2_{N-1}, z^2_N, 0, \ldots, 0}_{2N_{max}+1}\Big\}.$$

In Figure 1, we demonstrate the behavior of the error function on the truncation order of equation (3.11). This shows an essential feature of using the normalizing factor (3.10). It is clearly seen that the decrease has the same behavior both for the equation with and without the normalizing factor up to a certain number N. In the given example, this value is $N = 11$, but it should be taken into account that this number depends on the initial parameters of the problem. However, after this truncation number, the error of the solution without the weighting factor stops decreasing and starts increasing, while in the presence of the weighting factor, the error reaches the machine accuracy of calculations. This result provides a useful reference for future research.

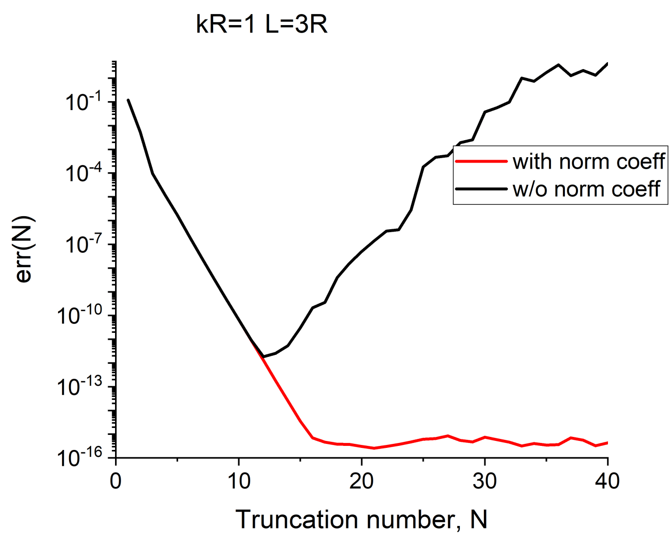

Figure 1: Error as a function of the truncation number N.

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. For sale by the Superintendent of Documents. National Bureau of Standards Applied Mathematics Series, No. 55. U. S. Government Printing Office, Washington, DC, 1964.

[2] J. D. Achenbach, *Wave Propagation in Elastic Solids*. Second printing of the 1973 first edition. North-Holland Series in Applied Mathematics and Mechanics, 16. North-Holland Publishing Co., Amsterdam, 1976.

[3] A. Amornwongpaibun, H. Luo and V. W. Lee, Scattering of anti-plane (SH) waves by a shallow semi-elliptical hill with a concentric elliptical tunnel. *Journal of Earthquake Engineering* **20** (2016), no. 3, 363–382.

[4] X. Antoine, A. Bendali and M. Darbas, Study and improvement of the condition number of the electric field integral equation. *Seventh Zaragoza-Pau Conference on Applied Mathematics and Statistics (Spanish) (Jaca, 2001)*, 73–80, Monogr. Semin. Mat. García Galdeano, 27, *Univ. Zaragoza, Zaragoza*, 2003.

[5] T. Balendra, D. P. Thambiratnam, Ch. G. Koh and S.-L. Lee, Dynamic response of twin circular tunnels due to incident SH-waves. *Earthquake Engineering & Structural Dynamics* **12** (), no. 2, 181–201.

[6] P. V. Bayly, P. G. Massouros, E. Christoforou, A. Sabet and G. M. Genin, Magnetic resonance measurement of transient shear wave propagation in a viscoelastic gel cylinder. *J. Mech. Phys. Solids* **56** (2008), no. 5, 2036–2049.

[7] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory.* Reprint of the 1983 original [MR0700400]. Classics in Applied Mathematics, 72. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.

[8] D. Felbacq, G. Tayeb and D. Maystre, Scattering by a random set of parallel cylinders. *Journal of the Optical Society of America A* **11** (1994), no. 9, 2526–2538.

[9] U. Hamhaber, F. A. Grieshaber, J. H. Nagel and U. Klose, Comparison of quantitative shear wave MR-elastography with mechanical compression tests. *Magnetic Resonance in Medicine* **49** (2003), no. 1, 71–77.

[10] Y. M. A. Hashash, J. J. Hook, B. Schmidt and J. I-Ch. Yao, Seismic design and analysis of underground structures. *Tunnelling and Underground Space Technology* **16** (2001), no. 4, 247–293.

[11] J. Henrych and G. R. Abrahamson, The dynamics of explosion and its use. *J. Appl. Mech.* **47** (1980), no. 1, p. 218.

[12] D. O. Herasymova, S. V. Dukhopelnykov, D. M. Natarov, T. L. Zinenko, M. Lucido and A. I. Nosich, Threshold conditions for transversal modes of tunable plasmonic nanolasers shaped as single and twin graphene-covered circular quantum wires. *Nanotechnology*, Volume **33** (2022), no. 49, 11 pp.

[13] H. F. Kara, A note on response of tunnels to incident SH-waves near hillsides. *Soil Dynamics and Earthquake Engineering* **90** (2016), 138–146.

[14] V. W. Lee and M. D. Trifunac, Response of tunnels to incident SH-waves. *Journal of the Engineering Mechanics Division* **105** (1979), no. 4, 643–659.

[15] Y. S. Li, T. B. Li and X. Zhang, Response of shallow-buried circular lining tunnel to incident P wave. *Applied Mechanics and Materials* **160** (2012), 331–336.

[16] D. K. Liu, B. Z. Gai and G. Y. Tao, Applications of the method of complex functions to dynamic stress concentrations. *Wave Motion* **4** (1982), no. 3, 293–304.

[17] Q. Liu and R. Wang, Dynamic response of twin closely-spaced circular tunnels to harmonic plane waves in a full space. *Tunnelling and Underground Space Technology* **32** (2012), 212–220.

[18] R. Mallet, *Great Neapolitan Earthquake of 1857. The First Principles of Observational Seismology as Developed in the Report to the Royal Society of London of the Expedition Made by Command of the Society into the Interior of the Kingdom of Naples, to Investigate the Circumstances of the Great Earthquake of December 1857*, Vol. 1. Chapman and Hall, London, 1862.

[19] R. Mallet, *Great Neapolitan Earthquake of 1857. The First Principles of Observational Seismology as Developed in the Report to the Royal Society of London of the Expedition Made by Command of the Society into the Interior of the Kingdom of Naples, to Investigate the Circumstances of the Great Earthquake of December 1857*, Vol. 2. Chapman and Hall, London, 1862.

[20] D. M. Natarov, R. Sauleau, M. Marciniak and A. I. Nosich, Effect of periodicity in the resonant scattering of light by finite sparse configurations of many silver nanowires. *Plasmonics* **9** (2014), 389–407.

[21] Y.-H. Pao and C.-C. Mow, *Diffraction of Elastic Waves and Dynamic Stress Concentrations.* Crane, Russak & Co. Inc., New York, 1973.

[22] H.-Y. She, L.-W. Li, S. J. Chua, W.-B. Ewe, O. J. F. Martin and J. R. Mosig, Enhanced backscattering by multiple nanocylinders illuminated by TE plane wave. *J. Appl. Phys.* **104** (2008), Article no. 064310.

[23] P. M. Shearer, *Introduction to Seismology.* Cambridge University Press, Cambridge, New York, 2009.

[24] C. Smerzini, J. Avilés, R. Paolucci and F. J. Sánchez-Sesma, Effect of underground cavities on surface earthquake ground motion under SH wave propagation. *Earthquake Engineering and Structural Dynamics* **38** (2009), no. 12, 1441–1460.

[25] A. Sommerfeld, Die greensche funktion der schwingungslgleichung. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **21** (1912), 309–352.

[26] A. Sommerfeld, *Partial Differential Equations in Physics*, Vol. 1. Academic Press Inc., New York, 1949.

[27] D.-H. Tsaur and K.-H. Chang, SH-waves scattering from a partially filled semi-circular alluvial valley. *Geophysical Journal International* **173** (2008), no. 1, 157–167.

[28] D.-H. Tsaur and M.-S. Hsu, SH waves scattering from a partially filled semi-elliptic alluvial valley. *Geophysical Journal International* **194** (2013), no. 1, 499–511.

[29] V. Twersky, On a multiple scattering theory of the finite grating and the Wood anomalies. *J. Appl. Phys.* **23** (1952), 1099–1118.

[30] Y. Xiao, Z.-lin Yang, Y. Yang, M.-han Sun and H.-yu Deng, Scattering of SH waves by orthotropic lining groups in half space. *Archive of Applied Mechanics* **92** (2022), 691–712.

[31] H. Xu, T. Li, J. Xu and Y. Wang, Dynamic response of underground circular lining tunnels subjected to incident P waves. *Mathematical Problems in Engineering* **2014**, Article ID 297424, 11 pp.

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