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THE AVERAGING METHOD FOR OPTIMAL CONTROL PROBLEMS OF INTEGRO-DIFFERENTIAL SYSTEMS ON THE HALF-AXIS

Abstract. For the optimal control problem of an integro-differential system with rapidly oscillating coefficients on the infinite interval, the convergence of optimal controls, trajectories, and quality criterion from the original problem to the corresponding triple of solutions of the averaged problem is established.

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რეზიუმე. უსასრულო ინტერვალზე სწრაფად რხევადი კოეფიციენტების მქონე ინტეგრო-დიფერენციალური სისტემის ოპტიმალური მართვის ამოცანისთვის დადგენილია ოპტიმალური მართვის, ტრაექტორიების და ხარისხის კრიტერიუმის კრებადობა საწყისი ამოცანიდან გასაშუალოებული ამოცანის ამონახსნების შესაბამის სამეულზე გადასვლისას.

1 Introduction

In this paper, the averaging method is applied to the optimal control problem for systems of integro-differential equations with rapidly oscillating coefficients and a small parameter on the half-axis.

The averaging method for practical problems was already applied by Newton in 1682. This method was justified for ordinary differential equations by Bogolyubov and Krylov [11]. For systems of integro-differential equations, results were obtained in [3, 4], for boundary value problems for such systems in [21], and for impulsive systems in [18, 19].

Taking into account that the averaged system usually has a much simpler structure, the averaging method is widely applied to the study of various optimal control problems. Results concerning the application of the averaging method to optimal control problems for ordinary differential equations are presented in [17, 20], and for functional-differential equations in [10]. For impulsive optimal control problems such results were obtained in [8, 9]. In addition, optimal control problems governed by integral equations with state and control constraints were studied in [15], where a version of Pontryagin's Maximum Principle was established.

In general, systems of integro-differential equations are mathematical models of many processes in natural sciences. Equations of this type arise as mathematical models of various processes in the natural sciences, in particular, in population dynamics [2], chemical kinetics, and hydrodynamics [1, 22]. They often include control that should minimize certain functionals related to the dynamics of processes. In this paper, the averaging method is applied to a nonlinear optimal control problem for a Volterra-type integro-differential system and a problem linear with respect to the control on the half-axis. On a finite interval, such problems were considered in [13].

In this paper, we show that the optimal trajectories and controls of the averaged problem converge to the corresponding solutions of the original problem, with convergence uniform over the set of admissible controls. This allows a significant simplification of the analysis of the optimal control problem without losing the essential characteristics of its dynamics. In addition, a connection between the problem on the half-axis and the corresponding averaged problem on a finite interval is established.

The paper consists of an introduction and two sections. Section 2 gives the problem statement for the nonlinear and linear cases. The main results are formulated and proved in Section 3.

2 Problem statement

In this section, we state the problem both in the nonlinear case and in the case linear with respect to the control.

2.1 Nonlinear optimal control problem for integro-differential equations with rapidly oscillating coefficients on the half-axis

Let us consider a nonlinear optimal control problem on the half-axis with a small parameter and rapidly oscillating coefficients:

$$\begin{cases} \dot{x} = X\left(\dfrac{t}{\varepsilon}, x(t), \displaystyle\int_0^t \varphi(t,s,x(s))\,ds, u(t)\right), \\ x(0,u(0)) = x_0, \end{cases} \tag{2.1}$$

where the cost function is given by

$$J_\varepsilon[u] = \int_0^\infty e^{-\gamma t} L(t, x(t), u(t))\,dt \longrightarrow \inf, \tag{2.2}$$

where $\varepsilon > 0$ is a small parameter, $\gamma > 0$ is a fixed constant characterizing the discount, x is a phase vector from $\mathbb{R}^d$, $u(t)$ is an m-dimensional control vector taking values in some set $U \subset \mathbb{R}^m$, $t \geq 0$.

Denote

$$\varphi_1(t,x) = \int_0^t \varphi(t,s,x)\,ds.$$

Assume that the following limit exists uniformly with respect to $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^m$:

$$\lim_{\varepsilon\to 0} \int_0^t \Big[X\Big(\frac{\tau}{\varepsilon}, x, \varphi_1(\tau,x), u\Big) - X_0(x,u)\Big]\,d\tau = 0 \tag{2.3}$$

for $t \geq 0$.

To the optimal control problems on the semi-axis (2.1), (2.2) with rapidly oscillating coefficients, we associate a simpler averaged control problem

$$\begin{cases} \dot{\xi} = X_0(\xi, u(t)), \\ \xi(0, u(0)) = x_0, \end{cases} \tag{2.4}$$

with the cost function

$$J[u] = \int_0^\infty e^{-\gamma t} L(t, \xi(t), u(t))\,dt \longrightarrow \inf. \tag{2.5}$$

In what follows, $|\cdot|$ denotes the norm of a vector in a finite-dimensional Euclidean space, and $\|\cdot\|$ denotes the matrix norm consistent with the vector norm.

For problem (2.1), (2.2) and its averaged problem (2.4), (2.5), we assume that the following conditions hold.

Condition 2.1. Admissible controls are m-dimensional vector functions $u(t)$ which, for almost all $t \geq 0$, take values in some compact set $U \subset \mathbb{R}^m$, and $u(\cdot)$ for each $T > 0$ belongs to a compact set U_T in $L^p(0,T)$ for some $p \geq 1$.

Condition 2.2. The function $X(t,x,y,u)$ is defined and continuous with respect to all variables in the domain $Q_0 = \{t \geq 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^n,\ u \in U \subset \mathbb{R}^m\}$, and the following conditions are satisfied:

(1) $X(t,x,y,u)$ is bounded in Q_0, i.e., there exists a constant $M > 0$ such that $|X(t,x,y,u)| \leq M$ for all $(t,x,y,u) \in Q_0$.

(2) $X(t,x,y,u)$ satisfies the Lipschitz condition in Q_0 with respect to $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^m$, with constant λ:

$$\big|X(t,x,y,u) - X(t,x_1,y_1,u_1)\big| \leq \lambda\big(|x-x_1| + |y-y_1| + |u-u_1|\big)$$

for all $(t,x,y,u), (t,x_1,y_1,u_1) \in Q_0$.

Condition 2.3. The function $\varphi(t,s,x)$ is defined and continuous in the domain $Q_1 = \{t \geq 0,\ s \geq 0,\ x \in \mathbb{R}^d\}$ and satisfies linear growth and a Lipschitz condition with respect to x. That is, there exists a constant $L_\varphi > 0$ such that $|\varphi(t,s,x) - \varphi(t,s,x_1)| \leq L_\varphi |x - x_1|$, $|\varphi(t,s,x)| \leq L_\varphi(1+|x|)$.

Condition 2.4. The limit in (2.3) exists uniformly with respect to $x \in \mathbb{R}^d$ and $u \in U$.

Condition 2.5. The scalar function $L(t,x,u)$ is defined and continuous with respect to all its arguments in the domain $Q_2 = \{t \geq 0,\ x \in \mathbb{R}^d,\ u \in U\}$, and satisfies a linear growth condition with respect to x and u in Q_2 with constant $M > 0$; specifically,

$$|L(t,x,u)| \leq M\big(1 + |x| + |u|\big).$$

2.2 Linear control problem

We also consider optimal control problem, linear with respect to the control for an integro-differential system on the semi-axis with a small parameter and rapidly oscillating coefficients:

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x(t), \int_0^t \varphi(t,s,x(s))\, ds\right) + f_1(x(t))u(t), \tag{2.6}$$
$$x(0) = x_0,$$

with the cost function

$$J_\varepsilon[u] = \int_0^\infty \left[e^{-jt}(A(t,x(t)) + B(t,u(t)))\right] dt \longrightarrow \inf, \tag{2.7}$$

where $\varepsilon > 0$ is a small parameter, $j > 0$ is a fixed constant characterizing the discount, x is a phase vector in $\mathbb{R}^d$, and $u(t)$ is an m-dimensional control vector taking values in a set $U \subset \mathbb{R}^m$.

For this problem, we also apply the averaging method, similarly to the nonlinear case. Denote

$$\varphi_1(t,x) = \int_0^t \varphi(t,s,x)\, ds.$$

Assume that the following limit exists uniformly with respect to $x \in \mathbb{R}^d$ for each $t \geq 0$:

$$\lim_{\varepsilon \to 0} \int_0^t \left[f\left(\frac{\tau}{\varepsilon}, x, \varphi_1(\tau,x)\right) - f_0(x)\right] d\tau = 0. \tag{2.8}$$

The optimal control problems (2.6), (2.7) with rapidly oscillating coefficients on the semi-axis correspond to a simpler averaged control problem

$$\dot{\xi} = f_0(\xi) + f_1(\xi)u(t), \;\; \xi(0,u(0)) = x_0, \tag{2.9}$$

with the cost function

$$J_0[u] = \int_0^\infty \left[e^{-jt}(A(t,\xi(t)) + B(t,u(t)))\right] dt \longrightarrow \inf. \tag{2.10}$$

For problem (2.6), (2.7) and the corresponding averaged problem (2.9), (2.10), we impose the following conditions.

Condition 2.6. Admissible controls are m-dimensional vector functions $u(\cdot) \in L^2(0,\infty)$ taking values in a closed, convex set $V \subset \mathbb{R}^m$; we also assume that $0 \in V$.

Condition 2.7. The function $f(t,x,y)$ is defined and continuous with respect to all its arguments in the domain $Q_3 = \{t \geq 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^n\}$, and the $n \times m$-dimensional matrix $f_1(x)$ is defined for $x \in \mathbb{R}^d$. Moreover, the following hold:

(1) $f(t,x,y)$ and $f_1(x)$ are bounded in their respective domains by a constant $M > 0$;

(2) $f(t,x,y)$ and $f_1(x)$ satisfy a Lipschitz condition with respect to x in their respective domains with a Lipschitz constant $\lambda > 0$.

Condition 2.8. The function $\varphi(t,s,x)$ is defined and continuous in the domain $Q_4 = \{t \geq 0,\ s \geq 0,\ x \in \mathbb{R}^d\}$ and satisfies linear growth and a Lipschitz condition with respect to x; that is, there exists a constant $L_\varphi > 0$ such that $|\varphi(t,s,x) - \varphi(t,s,x_1)| \leq L_\varphi|x - x_1|$, $|\varphi(t,s,x)| \leq L_\varphi(1 + |x|)$.

Condition 2.9. The limit in (2.8) exists uniformly with respect to $x \in \mathbb{R}^d$.

Condition 2.10. The scalar functions $A(t,x)$, $B(t,u)$, and $\frac{\partial B}{\partial u}(t,u)$ are defined for $t \geq 0$, $x \in \mathbb{R}^d$, $u \in V$, and continuous with respect to all their arguments. Moreover:

(1) $A(t,x) \geq 0$ and satisfies a linear growth condition in $x \in \mathbb{R}^d$ with a constant M, i.e., $|A(t,x)| \leq M(1+|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$;

(2) there exist the constants $a > 0$ and $a_1 > 0$ such that $a_1|u|^2 \geq B(t,u) \geq a|u|^2$ for all $t \geq 0$, $B(t,u)$ is convex with respect to $u \in V$, and there exists $a_2 > 0$ such that

$$\left|\frac{\partial B}{\partial u}(t,u)\right| \leq a_2|u|.$$

3 Main results

In this section, we present the main results and their proofs for both the nonlinear and linear cases.

3.1 Nonlinear Case

First, we note that it follows from Conditions 2.1, 2.2 and Theorem 3.1 [16] that for each admissible control $u(t)$, the solution $x(t,u)$ of the Cauchy problem (2.1) exists, is unique on $[0,\infty)$, and is an absolutely continuous function. Moreover, the solution of problem (2.1) satisfies the estimate

$$|x(t)| \leq |x_0| + Mt. \tag{3.1}$$

Hence, for the cost function (2.2), taking into account (3.1) and Condition 2.5, we obtain

$$|J_\varepsilon[u]| \leq \int_0^\infty e^{-\gamma t} M\big(1+|x(t)|+|u(t)|\big)\,dt \leq \int_0^\infty e^{-\gamma t} M\big(1+|x_0|+Mt\big)\,dt + C\int_0^\infty e^{-\gamma t}|u|\,dt < \infty$$

for some constant $C > 0$.

Thus, the cost function (2.2) is well-defined for all admissible controls.

From Condition 2.4 it follows that similar conclusions hold for the averaged problem.

The following lemma will be needed later, it guarantees the convergence of the solutions of the exact system (2.1) to the corresponding solutions of the averaged system (2.4). Let $T > 0$ be fixed.

Lemma 3.1. *Assume that Conditions 2.2–2.4 hold. Then, if $u_\varepsilon \to u_0$ as $\varepsilon \to 0$ in the norm of the space $L^p(0,T)$, the solution of the Cauchy problem (2.1) satisfies*

$$x_\varepsilon(t) \rightrightarrows \xi_0(t), \quad \varepsilon \to 0,$$

on $[0,T]$, where $\xi_0(t)$ is the solution of the Cauchy problem corresponding to $u = u_0$.

Proof. In addition to systems (2.1) and (2.4), consider the auxiliary system

$$\dot{z}_\varepsilon = X\left(\frac{t}{\varepsilon}, z_\varepsilon(t), \int_0^t \varphi(t,s,z_\varepsilon(s))\,ds, u_0\right), \quad z_\varepsilon(0,u_0(0)) = x_0. \tag{3.2}$$

Then

$$\begin{aligned} &|z_\varepsilon(t) - x_\varepsilon(t)| \\ &\quad \leq \int_0^t \left| X\left(\frac{s}{\varepsilon}, z_\varepsilon(s), \int_0^s \varphi(s,\tau,z_\varepsilon(\tau))\,d\tau, u_0(s)\right) - X\left(\frac{s}{\varepsilon}, x_\varepsilon(s), \int_0^s \varphi(s,\tau,x_\varepsilon(\tau))\,d\tau, u_\varepsilon(s)\right)\right| ds \end{aligned}$$

$$\leq \lambda \int\limits_0^t \left(|z_\varepsilon(s) - x_\varepsilon(s)| + L_\varphi \int\limits_0^s |z_\varepsilon(\tau) - x_\varepsilon(\tau)|\, d\tau \right) ds + \lambda \int\limits_0^T |u_\varepsilon(s) - u_0|\, ds$$

$$= \lambda \int\limits_0^t |z_\varepsilon(s) - x_\varepsilon(s)|\, ds + \lambda L_\varphi \int\limits_0^t \int\limits_0^s |z_\varepsilon(\tau) - x_\varepsilon(\tau)|\, d\tau\, ds + \lambda \int\limits_0^T |u_\varepsilon(s) - u_0|\, ds.$$

By the generalized Gronwall–Bellman lemma,

$$|z_\varepsilon(t) - x_\varepsilon(t)| \leq \lambda T^{\frac{1}{q}} \|u_\varepsilon - u_0\|_{L^p} \exp\big(\lambda T(1 + L_\varphi T)\big).$$

Furthermore, by Lemma 3.1 [13], it follows that $z_\varepsilon(t)$ converges to $\xi_0(t)$ as $\varepsilon \to 0$ uniformly on $[0, T]$.

Note that replacing the class $L^2(0, T)$ with $L^p(0, T)$ does not affect the proof, since any function in $L^p(0, T)$, $p \geq 1$, can be approximated arbitrarily closely by piecewise constant functions. This completes the proof. □

Remark. It follows from this theorem and the uniqueness of the Cauchy problem solution that if $u_\varepsilon \to u_0(t)$ as $\varepsilon \to 0$ in the norm $L^p(0, T)$ for each $T > 0$, then $x_\varepsilon(t)$ converges to $\xi_0(t)$ uniformly on every interval $[0, T]$. Hence, $x_\varepsilon(t) \to \xi_0(t)$ as $\varepsilon \to 0$ for any $t \geq 0$. Therefore, in this case, we obtain a pointwise convergence of the solutions of the original problem to the corresponding solutions of the averaged one.

The next theorem establishes a connection between optimal controls, optimal trajectories, and cost functionals of the exact problem (2.1), (2.2) and the averaged problem (2.4), (2.5).

Denote $J_\varepsilon^* = \inf\limits_u J_\varepsilon[u]$, $J_0^* = \inf\limits_u J_0[u]$, where the infimum is taken over all admissible controls.

Theorem 3.1. *Suppose that Conditions 2.1–2.5 hold, and there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, problems (2.1), (2.2) and (2.4), (2.5) have solutions $(x_\varepsilon^*(t), u_\varepsilon^*(t))$ and $(\xi^*(t), u^*(t))$, respectively. Then the following variational relations hold:*

(1) *$J_\varepsilon^* \to J_0^*$ as $\varepsilon \to 0$;*

(2) *for each $\eta > 0$, there exists ε_0 such that for $0 < \varepsilon < \varepsilon_0$,*

$$|J_\varepsilon^* - J_\varepsilon[u^*]| < \eta,$$

i.e., the optimal control of the averaged problem is nearly optimal for the original one;

(3) *there exists a sequence $\varepsilon_n \to 0$, $n \to \infty$, such that*

$$x_{\varepsilon_n}^*(t) \to \xi^*(t), \tag{3.3}$$

uniformly on every interval $[0, T]$, $T > 0$, and

$$`u_{\varepsilon_n}^*(t) \to u^*(t), \tag{3.4}$$

almost everywhere on $[0, \infty)$, and $u_{\varepsilon_n}^*(\,\cdot\,)$ converges to $u^*(\,\cdot\,)$ in the norm $L^p(0, T)$ for each $T > 0$.

If the averaged problem (2.4), (2.5) has a unique solution, then the convergences in (3.3), (3.4) hold for all $\varepsilon > 0$.

Proof. Let $\{\varepsilon_n\}_1^\infty$ be an arbitrary sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. By the assumptions of the theorem, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, problem (2.1), (2.2) has a solution $(x_{\varepsilon_n}^*(t), u_{\varepsilon_n}^*(t))$.

From Condition 2.1 it follows that there exists on $[0, 1]$ a subsequence $u_{\varepsilon_{n_1}}^*(\,\cdot\,)$ of the sequence $u_{\varepsilon_n}^*(\,\cdot\,)$ convergent in $L^p([0, 1])$. Let $u_1(\,\cdot\,)$ be its L^p-limit. Then there exists a subsequence of $u_{\varepsilon_{n_1}}^*(t)$ (denoted again by $u_{\varepsilon_{n_1}}^*(t)$) that converges pointwise to $u_1(t)$ everywhere on $[0, 1]$, except, possibly, on a set A_1 of Lebesgue measure zero.

By analogous arguments, from the sequence $u^*_{\varepsilon_{n_1}}(\,\cdot\,)$ one can extract on $[0,2]$ a subsequence $u^*_{\varepsilon_{n_2}}(\,\cdot\,)$ convergent in $L^p([0,2])$. Let $u_2(\,\cdot\,)$ be its L^p-limit. Then there exists a subsequence of $u^*_{\varepsilon_{n_2}}(t)$ (denoted again by $u^*_{\varepsilon_{n_2}}(t)$) that converges pointwise to $u_2(t)$ everywhere on $[0,2]$, except, possibly, on some set A_2 of Lebesgue measure zero.

Obviously, $u_1(t)=u_2(t)$ on the set $[0,1]\setminus(A_1\cup A_2)$.

Continuing this procedure on $[0,k]$, $k\in\mathbb{N}$, one can construct a subsequence $u^*_{\varepsilon_{n_k}}(\,\cdot\,)$ of the sequence $u^*_{\varepsilon_{n_{k-1}}}(\,\cdot\,)$ that converges to some function $u_k(\,\cdot\,)\in\mathcal{U}_k$ in the norm of $L^p([0,k])$. From it, one can also extract a subsequence, denoted by $u^*_{\varepsilon_{n_k}}$, which converges pointwise on $[0,k]$, except, possibly, on a set $A_k\subset[0,k]$ of measure zero. Obviously, $u_k(t)=u_{k-1}(t)$ on the set $[0,k-1]\setminus(\bigcup\limits_{i=1}^{k}A_i)$.

Denote $B=\bigcup\limits_{i=1}^{\infty}A_i$. Clearly, the set B has a Lebesgue measure zero. Let us construct a function $u_0(t)$ such that $u_0(t)=u_k(t)$ for $t\in[0,k]$. By construction, it has the following properties:

(a) $u_0(t)$ is defined on $[0,\infty)\setminus B$, that is, almost everywhere;

(b) $u_0(t)\in U$ for all $[0,\infty)\setminus B$;

(c) $u_0(\,\cdot\,)\in\mathcal{U}_T$ for every $T>0$.

Hence, $u_0(t)$ is an admissible control for problem (2.1), (2.2).

Using Cantor's diagonal method, one can construct a subsequence $u^*_{\varepsilon_{n_n}}(\,\cdot\,)$ of the original sequence $u^*_{\varepsilon_n}(\,\cdot\,)$ with the following properties:

(a) $u^*_{\varepsilon_{nn}}(\,\cdot\,)$ converges in the norm of $L^p([0,T])$ for every $T>0$ to $u_0(\,\cdot\,)$ as $n\to\infty$;

(b) $u^*_{\varepsilon_{n_n}}$ converges to $u_0(t)$ for all $t\in[0,\infty)\setminus B$.

Next, denote $u^*_{\varepsilon_{n_n}}(\,\cdot\,)$ by $u^*_{\varepsilon_m}$.

Then $x^*_{\varepsilon_m}(t)$ is the optimal trajectory of problem (2.1), (2.2). Let $\xi_0(t)$ denote the solution of the averaged problem (2.4) corresponding to the control $u_0(t)$.

From Lemma 3.1 it then follows that $x^*_{\varepsilon_m}(t)$ converges uniformly on each interval $[0,T]$, and pointwise on $[0,\infty)$ to $\xi_0(t)$ as $m\to\infty$.

Since $u^*_{\varepsilon_m}(t)$ are optimal controls and $x^*_{\varepsilon_m}(t)$ are optimal trajectories for problem (2.1), (2.2), we obtain

$$J^*_{\varepsilon_m}\le J^*_{\varepsilon_m}(u^*)=J^*_0+J_{\varepsilon_m}(u^*)-J_0(u^*). \tag{3.5}$$

But

$$|J_{\varepsilon_m}(u^*)-J_0(u^*)|\le\int\limits_0^\infty e^{-jt}\big|L(t,x_{\varepsilon_m}(t),u^*(t))-L(t,\xi^*(t),u^*(t))\big|\,dt. \tag{3.6}$$

Applying again Lemma 3.1 to the system

$$\dot{x}_{\varepsilon_m}=X\Big(\frac{t}{\varepsilon},x_{\varepsilon_m},u^*\Big),$$
$$\dot{\xi}^*=X_0(\xi^*,u^*),$$

we obtain a pointwise convergence, for each $t>0$, of $x_{\varepsilon_m}(t)$ to $\xi^*(t)$ as $\varepsilon_m\to0$.

Using the linear growth condition for L, estimate (3.1), and Condition 2.1 for the integrand in (3.6), we find an integrable majorant

$$e^{-jt}M\big(1+|x_0|+Mt+C\big), \tag{3.7}$$

where C is a constant characterizing the boundedness of the compact set U from Condition 2.1.

By the continuity of the function L, and by the Lebesgue dominated convergence theorem, a limiting transition is possible in (3.6), from which we obtain

$$J_{\varepsilon_m}(u^*)\to J_0(u^*),\ \ \varepsilon_m\to0. \tag{3.8}$$

On the other hand, since $u^*_{\varepsilon_m}$ is an admissible control for the averaged problem, we have

$$J^*_0 \le J_0(u^*_{\varepsilon_m}) = J^*_{\varepsilon_m} + J_0(u^*_{\varepsilon_m}) - J_{\varepsilon_m}(u^*_{\varepsilon_m}). \tag{3.9}$$

But

$$|J_0(u^*_{\varepsilon_m}) - J_{\varepsilon_m}(u^*_{\varepsilon_m})| \le |J_0(u^*_{\varepsilon_m}) - J_0(u_0)| + |J_{\varepsilon_m}(u^*_{\varepsilon_m}) - J_{\varepsilon_m}(u_0)| + |J_{\varepsilon_m}(u_0) - J_0(u_0)|. \tag{3.10}$$

Since

$$|J_0(u^*_{\varepsilon_m}) - J_0(u_0)| = \int\limits_0^\infty e^{-jt}\big|L(t, \xi_{\varepsilon_m}(t), u^*_{\varepsilon_m}(t)) - L(t, \xi_0(t), u_0(t))\big|\, dt,$$

where $\xi_{\varepsilon_m}(t)$ is the solution of the Cauchy problem

$$\dot\xi_{\varepsilon_m} = X_0(\xi_{\varepsilon_m}, u^*_{\varepsilon_m}), \quad \xi_{\varepsilon_m}(0, u_{\varepsilon_m}(0)) = x_0,$$

analogously to (3.8), we obtain

$$J_0(u^*_{\varepsilon_m}) \to J_0(u_0), \ \ \varepsilon_m \to 0. \tag{3.11}$$

It remains to show that

$$J_{\varepsilon_m}(u^*_{\varepsilon_m}) - J_{\varepsilon_m}(u_0) \to 0, \ \ \varepsilon_m \to 0. \tag{3.12}$$

Indeed,

$$|J_{\varepsilon_m}(u^*_{\varepsilon_m}) - J_{\varepsilon_m}(u_0)| \le \int\limits_0^\infty e^{-jt}\big|L(t, x^*_{\varepsilon_m}(t), u^*_{\varepsilon_m}(t)) - L(t, z_{\varepsilon_m}(t), u_0(t))\big|\, dt, \tag{3.13}$$

where z_{ε_m} is the solution of the Cauchy problem (3.2) with $\varepsilon = \varepsilon_m$.

Taking into account the almost everywhere convergence $u^*_{\varepsilon_m}$ to $u_0(t)$ as $\varepsilon_m \to 0$, and repeating the reasoning analogous to (3.8), we arrive at (3.12). It is also clear that

$$J_{\varepsilon_m}(u_0) - J_0(u_0) \longrightarrow 0, \ \ \varepsilon_m \to 0. \tag{3.14}$$

From (3.5), (3.8)–(3.14), we then obtain the estimates

$$J_{\varepsilon_m}(u^*_{\varepsilon_m}) - J^*_0 \le J^*_{\varepsilon_m} - J^*_0 \le J_{\varepsilon_m}(u^*) - J_0(u^*). \tag{3.15}$$

Hence,

$$J^*_{\varepsilon_m} \to J^*_0, \ \ \varepsilon_m \to 0.$$

Therefore, from any sequence $\{J^*_{\varepsilon_n}\}$ one can extract a subsequence converging to J^*_0. This proves the first assertion of the theorem.

It remains to verify that u_0 is the optimal control for the averaged problem, and that $y_0(t)$ is the corresponding optimal trajectory.

Analogously to the previous arguments, by the Lebesgue dominated convergence theorem, we have

$$J^*_{\varepsilon_m} = \int\limits_0^\infty e^{-jt} L(t, x^*_{\varepsilon_m}(t), u^*_{\varepsilon_m}(t))\, dt \longrightarrow \int\limits_0^\infty e^{-jt} L(t, \xi_0(t), u_0(t))\, dt, \ \ \varepsilon_m \to 0,$$

and hence, by (3.1), the pair $(u_0(t), \xi_0(t))$ is optimal, which completes the proof of statement 3) of the theorem.

To prove statement (2), we observe that

$$|J^*_\varepsilon - J_\varepsilon(u_0)| \le |J^*_\varepsilon - J^*_0| + |J_0(u_0) - J_\varepsilon(u_0)|.$$

Applying Lemma 3.1 to the systems

$$\dot x_\varepsilon = X\Big(\frac{t}{\varepsilon}, x_\varepsilon, u_0\Big), \quad \dot\xi_0 = X_0(\xi_0, u_0),$$

and arguing analogously to (3.8), we obtain $J_0(u_0) - J_\varepsilon(u_0) \to 0$, $\varepsilon \to 0$. Together with statement (1) of the theorem, this proves statement (2).

If the averaged problem (2.1), (2.2) has a unique solution, then from the above it follows that from any sequence $(x^*_{\varepsilon_m}, u^*_{\varepsilon_m})$ one can extract a convergent subsequence, and all such subsequences converge to the same limit. This completes the proof of the final statement of the theorem. □

We approximate the solutions of the optimal control problem (2.1) with the cost functional

$$I_\varepsilon[u] = \int_0^\infty e^{-\gamma t}(A(t, x(t)) + B(t, u(t)))\, dt \longrightarrow \inf \tag{3.16}$$

on the half-line by the solutions of the averaged control problems on the interval $[0, T]$, as $T \to \infty$.

Let $(x^*_\varepsilon(t), u^*_\varepsilon(t))$ be a solution of problem (2.1), (3.16) on the half-axis. Fix $T > 0$ and consider problem (2.4) with the cost function

$$I_{0T}[u] = \int_0^T e^{-\gamma t}(A(t, \xi(t)) + B(t, u(t)))\, dt \longrightarrow \inf \tag{3.17}$$

on the interval $[0, T]$. By Theorem 2.1 [13], this problem has an optimal solution (ξ^*_T, u^*_T) on $[0, T]$.

For problem (2.4), define an admissible control on $[0, \infty)$ by

$$\overline{u}_{T,\infty}(t) = \begin{cases} u^*_T(t), & t \in [0, T], \\ 0, & t > T, \end{cases}$$

and let $\xi_{T,\infty}$ denote the corresponding admissible trajectory.

Let $I^*_{0T} = \inf\limits_u I_{0T}[u]$.

Theorem 3.2. *Let Conditions 2.1–2.4 and 2.10 be satisfied. Then*

(1) $|I^*_\varepsilon - I^*_{0T}| \to 0$ *as* $\varepsilon \to 0$, $T \to \infty$*;*

(2) *there exist the sequences* $T_n \to \infty$, $\varepsilon_n \to 0$ *such that for any* $t > 0$ *we have*

$$|x^*_{\varepsilon_n}(t) - \xi_{T_n,\infty}(t)| \longrightarrow 0, \;\; T_n \to \infty, \;\; \varepsilon_n \to 0. \tag{3.18}$$

If the averaged problem (2.4), (3.17) has a unique solution, then the convergence in (3.18) holds for all $T \to \infty$ *and* $\varepsilon \to 0$.

Proof. We show that $\forall\, \eta > 0$ there exist T_0 and $\varepsilon_0 > 0$ such that $\forall\, T > T_0$, $\forall\, \varepsilon < \varepsilon_0$:

$$|I^*_{0T} - I^*_\varepsilon| < \eta.$$

Indeed, we have

$$|I^*_{0T} - I^*_\varepsilon| \le |I^*_{0T} - I^*_0| + |I^*_0 - I^*_\varepsilon|.$$

The first term does not depend on ε and tends to 0, as $T \to \infty$, by Theorem 2.5 [12]. The convergence of the second term to zero, as $\varepsilon \to 0$, follows from item (1) of Theorem 3.1.

(2) The proof of this item follows from the inequality

$$|x^*_{\varepsilon_n}(t) - \xi_{T_n,\infty}(t)| \le |x^*_{\varepsilon_n}(t) - \xi^*(t)| + |\xi^*(t) - \xi_{T_n,\infty}(t)|.$$

The convergence of the first term to zero as $\varepsilon_n \to 0$, for each fixed $t > 0$, follows from Theorem 3.1. The convergence $|\xi^*(t) - \xi_{T_n,\infty}(t)| \to 0$ as $T_n \to \infty$ follows from item (iv) of Theorem 2.5 [12]. □

3.2 Linear Case

Note that, by Conditions 2.7, 2.8 and Theorem 3.1 [16], it follows that for each admissible control $u(t)$ the solution $x(t,u)$ of the Cauchy problem (2.6) exists, is unique on $[0,\infty)$, and is an absolutely continuous function.

Moreover, for the solution of problem (2.6), similarly to the nonlinear case, for each $t>0$, we have the following estimate:

$$|x(t)| \le |x_0| + Mt + Mt^{\frac{1}{2}}\|u\|_{L^2[0,\infty)}. \tag{3.19}$$

Thus, for the cost function (2.7), taking into account (3.19) and item 2) of Condition 2.10, we obtain

$$|J_\varepsilon[u]| \le \int_0^\infty e^{-jt}M(1+|x(t)|)\,dt + \int_0^\infty B(t,u(t))\,dt$$
$$\le \int_0^\infty e^{-jt}M\big(1+|x_0|+Mt+Mt^{\frac{1}{2}}\|u\|_{L^2}\big)\,dt + a_1\|u\|_{L^2}^2 < \infty.$$

Therefore, the cost function (2.7) is well-defined for all admissible controls.

It follows from Condition 2.9 that similar conclusions also hold for the averaged problems.

For problems (2.6), (2.9), one can prove a theorem on the convergence of solutions of the original problem (2.6) to the corresponding solutions of the averaged system (2.9) on the half-axis, analogous to Theorem 3.1.

Lemma 3.2. *Let Conditions 2.7–2.9 be satisfied. Then, if $u_\varepsilon \to u_0$ as $\varepsilon \to 0$ in the norm of the space $L^p(0,T)$, where $T>0$, the solution of the Cauchy problem (2.1) satisfies*

$$x_\varepsilon(t) \rightrightarrows \xi_0(t), \quad \varepsilon \to 0,$$

on $[0,T]$, where $\xi_0(t)$ is the solution of the Cauchy problem with $u=u_0$.

The following theorem describes the relationship between the optimal controls, the optimal trajectories, and the cost functionals of the original problem (2.6), (2.7) and the averaged problem (2.9), (2.10).

Denote $J_\varepsilon^* = \inf_u J_\varepsilon[u]$, $J_0^* = \inf_u J_0[u]$, where the infimum is taken over all admissible controls.

Theorem 3.3. *Let Conditions 2.6–2.10 be satisfied. Then problems (2.6), (2.7) and (2.9), (2.10) have solutions $(x_\varepsilon^*(t), u_\varepsilon^*(t))$ and $(\xi^*(t), u^*(t))$, respectively.*

Moreover, the following variational relations hold:

(1) *$J_\varepsilon^* \to J_0^*$ as $\varepsilon \to 0$;*

(2) *for each $\eta>0$, there exists $\varepsilon_0=\varepsilon_0(\eta)$ such that for all $0<\varepsilon<\varepsilon_0$ we have $|J_\varepsilon^* - J_\varepsilon[u^*]| < \eta$;*

(3) *there exists a sequence $\varepsilon_n \to 0$, $n\to\infty$, such that*

$$x_{\varepsilon_n}^*(t) \to \xi(t) \tag{3.20}$$

uniformly on each interval $[0,T]$ for any $T>0$, and

$$u_{\varepsilon_n}^* \rightharpoonup u^* \tag{3.21}$$

in $L^2(0,\infty)$.

If the averaged problem (2.6), (2.7) has a unique solution, then the convergences in (3.20) and (3.21) hold for all $\varepsilon \to 0$.

The proof of the theorem is entirely analogous to that of Theorem 2.3 in [7].

We approximate the solutions of the optimal control problem (2.6), (2.7) on the half-line by the solutions of the averaged control problems on the interval $[0,T]$ as $T\to\infty$.

Similarly to the nonlinear case, let $(x_\varepsilon^*(t),u_\varepsilon^*(t))$ be a solution of problem (2.6), (2.7) on the half-axis. Fix $T>0$ and consider problem (2.9) with the cost functional

$$I_{0T}[u]=\int_0^T e^{-jt}\big(A(t,\xi(t))+B(t,u(t))\big)\,dt\longrightarrow\inf$$

on the interval $[0,T]$. Then, by Theorem 2.2 [13], this problem has a solution (ξ_T^*,u_T^*) on $[0,T]$.

For problem (2.9), (2.10), define an admissible control on $[0,\infty)$ as follows:

$$\overline{u}_{T,\infty}(t)=\begin{cases}u_T^*(t), & t\in[0,T],\\ 0, & t>T,\end{cases}$$

and denote by $\xi_{T,\infty}$ the corresponding admissible trajectory.

Let $I_{0T}^*=\inf\limits_u I_{0T}[u]$.

Theorem 3.4. *Let Conditions 2.6–2.10 hold. Then:*

(1) $|I_\varepsilon^*-I_{0T}^*|\to 0$ *as* $\varepsilon\to0$, $T\to\infty$;

(2) *there exist the sequences* $T_n\to\infty$, $\varepsilon_n\to0$, *such that for any* $t>0$,

$$|x_{\varepsilon_n}^*(t)-\xi_{T_n,\infty}(t)|\longrightarrow 0,\quad T_n\to\infty,\quad \varepsilon_n\to0; \tag{3.22}$$

(3)

$$u_{\varepsilon_n}^*-\overline{u}_{T_n,\infty}\rightharpoonup 0\ \text{ as }\ \varepsilon_n\to0,\quad T_n\to\infty \tag{3.23}$$

in $L^p(0,\infty)$.

If the averaged problem (2.9), (2.10) *has a unique solution, then the convergences in* (3.22), (3.23) *hold for all* $T\to\infty$.

Proof. The proofs of the first two statements are analogous to that of statements (1), (2) in Theorem 3.2.

Let us prove statement (3), namely, let us show that for each $\eta>0$, there exist $\varepsilon_0>0$ and T_0 such that for every $\varepsilon_n<\varepsilon_0$, $T_n>T_0$, the following inequality holds:

$$\Big|\int_0^\infty u_{\varepsilon_n}^*\varphi(t)\,dt-\int_0^\infty \overline{u}_{T_n,\infty}\varphi(t)\,dt\Big|<\eta$$

for any $\varphi\in L^2(0,\infty)$.

Let u^* be the optimal control of the averaged problem on the half-axis. We have

$$\begin{aligned}\Big|\int_0^\infty u_{\varepsilon_n}^*\varphi(t)\,dt-\int_0^\infty \overline{u}_{T_n,\infty}\varphi(t)\,dt\Big|&\le\Big|\int_0^\infty u_{\varepsilon_n}^*\varphi(t)\,dt-\int_0^\infty u^*\varphi(t)\,dt+\int_0^\infty u^*\varphi(t)\,dt-\int_0^\infty \overline{u}_{T_n,\infty}\varphi(t)\,dt\Big|\\&\le\Big|\int_0^\infty (u_{\varepsilon_n}^*-u^*)\varphi(t)\,dt+\int_0^\infty (u^*-\overline{u}_{T_n,\infty})\varphi(t)\,dt\Big|.\end{aligned}$$

The weak convergence of the first term to 0 as $\varepsilon_n\to0$ follows from Theorem 3.1. The weak convergence of the second term to 0 in the last inequality follows from Theorem 2.5 [12] as $T_n\to\infty$. Note that this term does not depend on ε_n. □

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