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ASYMPTOTIC INTEGRATION OF SYSTEM OF SINGULARY PERTURBED DIFFERENTIAL EQUATIONS WITH UNSTABLE TURNING POINT

Abstract. This paper investigates the case where the principal matrix contains negative components that significantly affect the asymptotic behavior of solutions. Constructive conditions for the existence of an asymptotic solution to a system of singularly perturbed fourth-order differential equations with a differential turning point are established. An algorithm for constructing the corresponding approximate solution is proposed. Applying the method of essentially singular functions, an asymptotic representation of the solution is derived that reflects the specific structural features of the problem. Particular attention is given to the case where the spectrum of the limiting operator contains multiple eigenvalues and zero spectral elements. The analysis conducted provides a deeper understanding of the behavior of solutions at critical points and lays the foundation for further studies of related classes of problems.

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Key words and phrases. Asymptotic solution, singularly perturbed system of differential equations, turning point, essentially singular functions, space of resonance-free solutions, uniform asymptotics, singular point, perturbation.

რეზიუმე. ნაშრომში გამოკვლეულია შემთხვევა, როცა მთავარი მატრიცა შეიცავს უარყოფით კომპონენტებს, რომლებიც მნიშვნელოვნად მოქმედებს ამონახსნების ასიმპტოტურ ქცევაზე. დადგენილია ასიმპტოტური ამონახსნის არსებობის კონსტრუქციული პირობები სინგულარულად შეშფოთებული მეოთხე რიგის დიფერენციალური განტოლებების სისტემისთვის დიფერენციალური შემობრუნების წერტილით. შემოთავაზებულია შესაბამისი მიახლოებითი ამონახსნის აგების ალგორითმი. არსებითად სინგულარული ფუნქციების მეთოდის გამოყენებით, მიღებულია ამონახსნის ასიმპტოტური წარმოდგენა, რომელიც ასახავს ამოცანის სპეციფიკურ სტრუქტურულ მახასიათებლებს. განსაკუთრებული ყურადღება ეთმობა შემთხვევას, როდესაც ზღვარითი ოპერატორის სპექტრი შეიცავს ჯერად საკუთარ მნიშვნელობებს და ნულოვან სპექტრულ ელემენტებს. ჩატარებული ანალიზი იძლევა კრიტიკულ წერტილებში ამონახსნების ქცევის უფრო ღრმა გაგების საშუალებას და საფუძველს უყრის ამოცანათა მსგავსი კლასის შემდგომ კვლევას.

1 Introdution

Singularly perturbed differential equations are widely used in mathematical modeling of processes characterized by the presence of small parameters at higher derivatives. Such problems arise in many areas of science and technology, in particular in control theory, mechanics, physics, biology, and chemical kinetics. Of particular interest are the problems with turning points. At such points, the type of solution changes, which requires the construction of special internal asymptotics and the application of modified approaches. Given the specificity of the mathematical formulation, the study of the asymptotics of solutions of singularly perturbed systems of differential equations with differential turning points is a complex task that requires the use of special methods. It is important to note that the study of the asymptotics of solutions to singularly perturbed systems of differential equations with differential turning points is an area that is actively researched, with new methods being developed and new results are being provided that find application in various fields of applied mathematics [3, 4, 6, 8–10].

2 Statement of the problem

Let us consider a system of singularly perturbed differential equations with a differential turning point

$$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{2.1}$$

where $A(x,\varepsilon)$ has the structure

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1,$$

and $A_0(x)$ and A_1 form the matrices

$$A_0(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a(x), b(x), H(x) \in C^{\infty}[-l,0].$$

We consider the case where the coefficients of the main matrix have the same signs, that is, the following conditions are met:

C 1. $A_0(x)$, $H(x) \in C^{\infty}[-l,0]$.

C 2. $a(x) = x\widetilde{a}(x)$, $\widetilde{a}(x) < 0$, $b(x) < 0$.

When the conditions **C 1** and **C 2** are fulfilled, the system contains an unstable turning point, and the degenerate equation corresponding to the system does not have a smooth solution in the vicinity of the turning point. Therefore, it is not used explicitly to construct the third linearly independent solution for the system. To construct a uniform asymptotics of solution, the method of essentially singular functions is used [2]. The constructed solutions were evaluated as $\varepsilon \to 0$.

The characteristic equation corresponding to system (2.1) has the form

$$|A(x,0) - \lambda E| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ b(x) & a(x) & -\lambda \end{vmatrix} = -\lambda^3 + x\widetilde{a}(x)\lambda = 0.$$

Since we are investigating the unstable turning point $x = 0$, the roots of the characteristic equation are real numbers, i.e., $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{x\widetilde{a}(x)}$.

The degenerate equation for system (2.1) in this case will look like

$$-x\widetilde{a}(x)y'(x,\varepsilon) - b(x)y(x,\varepsilon) = h(x), \tag{2.2}$$

where $\varepsilon > 0$ is a small parametr, $\widetilde{a}(x), b(x), h(x) \in C^{\infty}[-l;0]$.

3 Space of non-resonant solutions

In a singularly perturbed system with a turning point near the independent variable x, we introduce a new vector variable $t = \varepsilon^{-p} \cdot \varphi(x)$. Then, instead of the desired vector function $Y(x, \varepsilon)$, a new "extended vector function" $\widetilde{Y}(x, t, \varepsilon)$ is studied. Moreover, the expansion is carried out in such a way that the condition is satisfied, as in the regularization method [2].

We select a set of functions in which the problem is expanded:

$$\widetilde{L}_\varepsilon \widetilde{Y_k}(x, t, \varepsilon) \equiv \mu\varphi' \frac{\partial \widetilde{y}(x, t, \varepsilon)}{\partial t} + \mu^3 \frac{\partial \widetilde{y}(x, t, \varepsilon)}{\partial x} - A(x, \varepsilon)\widetilde{Y_k}(x, t, \varepsilon) = H(x)$$

will be regularly perturbed with respect to a small parameter. Toward this end, we consider the sets (subspaces) of functions

$$D_{1k} = \alpha_{1k}(x, \varepsilon)U_1(t) + \varepsilon^\gamma \beta_{1k}(x, \varepsilon)U_1'(t), \quad D_{2k} = \alpha_{2k}(x, \varepsilon)U_2(t) + \varepsilon^\gamma \beta_{2k}(x, \varepsilon)U_2'(t),$$
$$D_{3k} = f_k(x, \varepsilon)\nu(t) + \varepsilon^\gamma g_k(x, \varepsilon)\nu'(t), \quad D_{4k} = \overline{\omega}_k(x, \varepsilon),$$

where $U_i(t)$ $(i = \overline{1, 2})$ are the Airy–Langer functions [2].

Subspaces D_{1k} and D_{2k} contain solutions of a homogeneous system of singularly perturbed differential equations, whose structure contains essentially special functions $U_i(t)$. The subspace D_{3k} contains solutions of a non-homogeneous system of singularly perturbed differential equations, whose structure contains an essentially special function $\nu(t)$ and its derivative [2]. The subspace D_{4k} contains solutions of a homogeneous and inhomogeneous system of singularly perturbed differential equations that correspond to the root of the characteristic equation $\lambda_1 = 0$, and does not contain any essentially special functions.

In this case, the turning point will be unstable, as in the previous case, but the solutions of the degenerate equation will not be sufficiently smooth at the point $x = 0$. If $b(x) < 0$ in (2.2), we get $\frac{b(0)}{\overline{a}(0)} = \rho > 0$ [1]. Therefore, we cannot repeat the logic and use the reasonings described in the case of [7], since the solution of the degenerate differential equation (2.1) and its derivatives are not sufficiently smooth at the point $x = 0$. This is due to the fact that the functions $\omega_{kr}(x)$ cannot be expanded in a series in powers of a small parameter in such a way that the coefficients of these expansions are continuous functions on the entire interval $[-l; 0]$. Therefore, it cannot be used to construct the third linearly independent solution of a homogeneous system of singularly perturbed differential equations [7]. So, the structure of the solution to problem (2.1) with this type of turning point cannot be reduced to the structures of solutions of those problems that were considered in previous cases. This is the main feature of problem (2.1) for the case with subtractive matrix coefficients. The difficulties that arise in constructing the third formal solution will be described below.

4 Regularization of the system of singularly perturbed equation

An element of this space has the form

$$\widetilde{Y_k}(x, t, \varepsilon) = \sum_{i=1}^{2} \left[\alpha_{ik}(x)U_i(t) + \beta_{ik}(x)U_i'(t)\right] + f_k(x)\nu(t) + \varepsilon^\gamma g_k(x)\nu'(t) + \omega_k(x).$$

For convenience, we introduce the notation $U_1(t) \equiv \mathrm{Ai(t)}$, $U_2(t) \equiv \mathrm{Bi(t)}$.

Let us write the result of the extended operator action $\widetilde{L}_\varepsilon$ on elements D_{1k} and D_{2k} in the form of two vector equations:

$$U_i'(t) : \alpha_{ik}(x, \varepsilon)\varphi'(x) - \left[A_0(x) + \mu^3 A_1\right]\beta_{ik}(x, \varepsilon) = -\mu^3 \beta_{ik}'(x, \varepsilon),$$
$$U_i(t) : \beta_{ik}(x, \varepsilon)\varphi(x)\varphi'(x) - \left[A_0(x) + \mu^3 A_1\right]\alpha_{ik}(x, \varepsilon) = -\mu^3 \alpha_{ik}'(x, \varepsilon).$$

From the vector equations, we uniquely determine the exponent $\mu = \varepsilon^{\frac{1}{3}}$:

$$\begin{cases} \alpha_{i1}(x,\varepsilon)\varphi'(x) = -\mu^3\left[\beta'_{i1}(x,\varepsilon) - \beta_{i2}(x,\varepsilon)\right], \\ \alpha_{i2}(x,\varepsilon)\varphi'(x) - \beta_{i3}(x,\varepsilon) = -\mu^3\beta'_{i2}(x,\varepsilon), \\ \alpha_{i3}(x,\varepsilon)\varphi'(x) + b(x)\beta_{i1}(x,\varepsilon) - a(x)\beta_{i2}(x,\varepsilon) = -\mu^3\beta'_{i3}(x,\varepsilon), \\ \varphi(x)\varphi'(x)\beta_{i1}(x,\varepsilon) = -\mu^3\left[\alpha'_{i1}(x,\varepsilon) - \alpha_{i2}(x,\varepsilon)\right], \\ \varphi(x)\varphi'(x)\beta_{i2}(x,\varepsilon) - \alpha_{i3}(x,\varepsilon) = -\mu^3\alpha'_{i2}(x,\varepsilon), \\ \varphi(x)\varphi'(x)\beta_{i3}(x,\varepsilon) + b(x)\alpha_{i1}(x,\varepsilon) - a(x)\alpha_{i2}(x,\varepsilon) = -\mu^3\alpha'_{i3}(x,\varepsilon), \end{cases} \tag{4.1}$$

which is a regularly perturbed system of relatively small parameter μ.

5 Construction of formal solutions of a homogeneous system

To construct the asymptotics of solutions of a homogeneous extended system (2.1), we will use a system of algebraic equations (4.1), whose unknown coefficients we will look for in the form of the series

$$\alpha_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty}\mu^r\alpha_{ikr}(x), \quad \beta_{ik}(x,\varepsilon) = \sum_{r=0}^{+\infty}\mu^r\beta_{ikr}(x). \tag{5.1}$$

To determine the components of vector functions $\alpha_{ikr} = colomn(\alpha_{i1r}(x), \alpha_{i2r}(x), \alpha_{i3r}(x))$ and $\beta_{ikr}(x) = colomn(\beta_{i1r}(x), \beta_{i2r}(x), \beta_{i3r}(x))$, we obtain the following recurrent systems of equations:

$$\Phi(x)Z_{k0}(x) = 0, \;\; r = 0,1,2, \quad \Phi(x)Z_{kr}(x) = FZ_{k(r-3)}(x), \;\; r \geq 3. \tag{5.2}$$

Here,

$$Z_{kr}(x) = colomn(\alpha_{i1r}(x), \alpha_{i2r}(x), \alpha_{i3r}(x), \beta_{i1r}(x), \beta_{i2r}(x), \beta_{i3r}(x)),$$

$$\Phi(x) = \begin{pmatrix} \varphi'(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi'(x) & 0 & 0 & 0 & -1 \\ 0 & 0 & \varphi'(x) & b(x) & -x\widetilde{a}(x) & 0 \\ 0 & 0 & 0 & \varphi(x)\varphi'(x) & 0 & 0 \\ 0 & 0 & -1 & 0 & \varphi(x)\varphi'(x) & 0 \\ b(x) & -x\widetilde{a}(x) & 0 & 0 & 0 & \varphi(x)\varphi'(x) \end{pmatrix}.$$

Let us calculate the determinant of this system. We have

$$\det\Phi(x) = (a^2(x) - 2a(x)\varphi(x)\varphi'^2(x) + \varphi^2(x)\varphi'^4(x)) \cdot \varphi(x)\varphi'^2(x).$$

We define the function $\varphi(x)$ as the solution to the problem

$$\varphi^2(x)\varphi'^4(x) + 2a(x)\varphi(x)\varphi'^2(x) + a^2(x) = 0,$$

which we can write in a simpler form

$$\varphi(x)\varphi'^2(x) = -a(x) \equiv -x\widetilde{a}, \;\; \varphi(0) = 0. \tag{5.3}$$

It should be noted that the solutions of the equation

$$U''(t) - tU(t) = 0,$$

that is, the functions Ai(t) and Bi(t), will be stable as $t \to -\infty$. So, we need the solution to problem (5.3) to be a function $\varphi(0) < 0$.

The solution to problem (5.3) under the condition $\varphi(0) = 0$ is the function

$$\varphi(x) = \left(\frac{3}{2}\int\limits_0^x \sqrt{-x\widetilde{a}(x)}\,dx\right)^{\frac{2}{3}}.$$

A regularizing function of this type was considered in [1,5]. Since $\det\Phi(x) \equiv 0$, there is a nontrivial solution to the homogeneous system (2.1) of the form

$$Z_{k0}(x) = colomn\Big(0, \frac{1}{\varphi'(x)}\,\beta_{i30}(x), -\varphi(x)\varphi'(x)\beta_{i20}(x), 0, \beta_{i20}(x), \beta_{i30}(x)\Big), \tag{5.4}$$

where $\beta_{0ik}(x)$, $i = 1, 2$, $i = \overline{1, 3}$, up to a certain moment, are arbitrary, fairly smooth functions with $x \in [-l; 0]$.

Let us solve nonhomogeneous systems (5.2). First, consider these systems for $r = 3$. Considering the obtained solution (5.4), we have

$$\begin{cases} \varphi'(x)\alpha_{i13}(x) = \beta_{i20}(x) - \beta'_{i10}(x) \equiv \beta_{i20}(x), \\ \varphi'(x)\alpha_{i23}(x) - \beta_{i33}(x) = -\beta'_{i20}(x), \\ \varphi'(x)\alpha_{i33}(x) + b(x)\beta_{i13}(x) + a(x)\beta_{i23}(x) = -\beta'_{i30}(x) \end{cases} \tag{5.5}$$

and

$$\begin{cases} \varphi(x)\varphi'(x)\beta_{i13}(x) = -\alpha'_{i10}(x) + \alpha_{i20}(x) \equiv \alpha_{i20}(x) \equiv [\varphi'(x)]^{-1}\beta_{i30}(x), \\ \varphi(x)\varphi'(x)\beta_{i23}(x) - \alpha_{i33}(x) = -\alpha'_{i20}(x) \equiv -\dfrac{d}{dx}\left([\varphi'(x)]^{-1}\beta_{i30}(x)\right), \\ \varphi\varphi'(x)\beta_{i33}(x) + b(x)\alpha_{i13}(x) + a(x)\alpha_{i23}(x) = -\alpha'_{i30}(x). \end{cases} \tag{5.6}$$

With this definition of vector functions $Z_{k0}(x)$, there are the solutions to inhomogeneous systems of algebraic equations (5.5) and (5.6) of the form

$$Z_{k3}(x) = colomn(z_{k13}, z_{k23}, z_{k33}, z_{k43}, z_{k53}, z_{k63}),$$

where

$$z_{i13} = \frac{1}{\varphi'(x)}\,\beta_{i20}(x), \quad z_{i23} = \frac{-\beta'_{i20}(x) + \beta_{i33}(x)}{\varphi'(x)},$$
$$z_{i33} = \frac{-\beta'_{i30}(x) - a(x)\beta_{i23}(x) - b(x)(\varphi(x))^{-1}(\varphi'(x))^{-2}\beta_{i30}}{\varphi'(x)},$$
$$z_{i43} = (\varphi(x))^{-1}(\varphi'(x))^{-2}\beta_{i20}(x), \quad z_{i53} = \beta_{i21}(x), \quad z_{i63} = \beta_{i31}(x),$$

where $\beta_{i21}(x)$ and $\beta_{i31}(x)$, as in (5.4), up to a certain moment, are arbitrary, sufficiently smooth functions for all $x \in [-l; 0]$.

6 Construction of formal solutions of a nonhomogeneous system

Formally, the solution of a homogeneous system (2.1) can be represented as rows (5.1):

$$Y_{ik}(x, \varepsilon^{-\frac{2}{3}}\varphi(x), \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \left[\alpha_{ikr}(x)U_i(\varepsilon^{-\frac{2}{3}}\varphi(x)) + \varepsilon^{\frac{1}{3}}\beta_{ikr}(x,\varepsilon)U'_i(\varepsilon^{-\frac{2}{3}}\varphi(x))\right]$$
$$+ \sum_{r=-2}^{\infty} \mu^r \left[f_{kr}(x)\nu(t) + \mu g_{kr}(x)\nu'(t) + \overline{\omega}_{kr}(x)\right]. \tag{6.1}$$

To determine the components of the vector functions

$$f_{kr}(x) = colomn(f_{1r}(x), f_{2r}(x), f_{3r}(x)), \quad g_{kr}(x) = colomn(g_{1r}(x), g_{2r}(x), g_{3r}(x)),$$

we replace the rows $f_{kr}(x)$ and $g_{kr}(x)$ into a regularized system of equations

$$\begin{cases} f_1(x,\varepsilon)\varphi'(x) = -\mu^3\big[g_1'(x,\varepsilon) - g_2(x,\varepsilon)\big], \\ f_2(x,\varepsilon)\varphi'(x) - g_3(x,\varepsilon) = -\mu^3 g_2'(x,\varepsilon), \\ f_3(x,\varepsilon)\varphi'(x) - b(x)g_{i1}(x,\varepsilon) - a(x)g_2(x,\varepsilon) = -\mu^3 g_3'(x,\varepsilon), \\ \varphi(x)\varphi'(x)g_1(x,\varepsilon) = -\mu^3[f_1'(x,\varepsilon) - f_2(x,\varepsilon)], \\ \varphi(x)\varphi'(x)g_2(x,\varepsilon) - f_3(x,\varepsilon) = -\mu^3 f_2'(x,\varepsilon), \\ \varphi(x)\varphi'(x)g_3(x,\varepsilon) + b(x)f_1(x,\varepsilon) + a(x)f_2(x,\varepsilon) = -\mu^3\alpha_{i3}'(x,\varepsilon). \end{cases}$$

From this system, we obtain the following systems of recurrent equations:

$$\Phi(x)\cdot Z_0^{part.}(x) = 0, \;\; r = -2,-1,0, \quad \Phi(x)\cdot Z_r^{part.}(x) = -Z_{r-3}^{part.}(x), \;\; r \geq 1. \tag{6.2}$$

In the resulting recursions (6.2), $\Phi(x)$ is the matrix and

$$Z_r^{part.}(x) = colomn\big(f_{1r}(x), f_{2r}(x), f_{3r}(x), g_{1r}(x), g_{2r}(x), g_{3r}(x)\big)$$

is an unknown vector function.

Recall that to construct the third formal solution of the homogeneous system (2.1), it is necessary to construct only partial solutions of this system. Consider the equation

$$\mu^3\omega_k'(x,\varepsilon) - A(x,\varepsilon)\omega_k(x,\varepsilon) = H(x) - \mu^2\varphi'(x)\pi^{-1}g_k(x,\varepsilon). \tag{6.3}$$

We represent equation (6.3) in the form of a system

$$\begin{cases} \mu^3\overline{\omega}_1'(x) = -\mu^2\varphi'(x)g_1(x) + \mu^3\overline{\omega}_2(x), \\ \mu^3\overline{\omega}_2'(x) - \overline{\omega}_3(x) = -\mu^2\varphi'(x)g_2(x), \\ \mu^3\overline{\omega}_3'(x) - b(x)\overline{\omega}_1(x) - a(x)\overline{\omega}_2(x) = h(x) - \mu^2\varphi'(x)g_3(x). \end{cases} \tag{6.4}$$

To determine the components of vector functions $\omega_k(x)$, let us substitute a series

$$\overline{\omega}_k(x,\varepsilon) = \sum_{r=0}^{+\infty}\mu^r\omega_{kr}(x)$$

into system (6.4). As a result, we obtain a system of the form

$$\begin{cases} \mu\big[\mu^0\overline{\omega}_{10}'(x) + \mu^1\overline{\omega}_{11}'(x) + \mu^2\overline{\omega}_{12}'(x) + \cdots\big] \\ \qquad = -\varphi'(x)\big[\mu^{-2}g_{1(-2)}(x) + \mu^{-1}g_{1(-1)}(x) + \mu^0 g_{10}(x) + \cdots\big] \\ \qquad\qquad + \mu\big[\mu^0\overline{\omega}_{20}(x) + \mu^1\overline{\omega}_{21}(x) + \mu^2\overline{\omega}_{22}(x) + \cdots\big], \\ \mu\big[\mu^0\overline{\omega}_{20}'(x) + \mu^1\overline{\omega}_{21}'(x) + \mu^2\overline{\omega}_{22}'(x) + \cdots\big] - \big[\mu^0\overline{\omega}_{30}(x) + \mu^1\omega_{31}(x) + \mu^2\overline{\omega}_{32}(x) + \cdots\big] \\ \qquad = -\mu^2\varphi'(x)\big[\mu^{-2}g_{2(-2)}(x) + \mu^{-1}g_{2(-1)}(x) + \mu^0 g_{20}(x) + \cdots\big], \\ \mu\big[\mu^0\overline{\omega}_{30}'(x) + \mu^1\overline{\omega}_{31}'(x) + \mu^2\omega_{32}'(x) + \cdots\big] - b(x)\big[\mu^0\overline{\omega}_{10}(x) + \mu^1\overline{\omega}_{11}(x) + \mu^2\overline{\omega}_{12}(x) + \cdots\big] \\ \qquad\qquad -a(x)\big[\mu^0\overline{\omega}_{20}(x) + \mu^1\overline{\omega}_{21}(x) + \mu^2\overline{\omega}_{22}(x) + \cdots\big] \\ \qquad = h(x) - \mu^2\varphi'(x)\big[\mu^{-2}g_{3(-2)}(x) + \mu^{-1}g_{3(-1)}(x) + \mu^0 g_{30}(x) + \cdots\big]. \end{cases} \tag{6.5}$$

First, let us consider the recurrent systems (6.2).

We write the system of recurrent equations (6.2) for $r = -2$:

$$\begin{cases} \varphi'(x)f_{1(-2)}(x,\varepsilon) = 0, \\ \varphi'(x)f_{2(-2)}(x,\varepsilon) - g_{i30}(x,\varepsilon) = 0, \\ \varphi'(x)f_{3(-2)}(x,\varepsilon) - b(x)g_{1(-2)}(x,\varepsilon) - a(x)g_{2(-2)}(x,\varepsilon) = 0, \\ \varphi(x)\varphi'(x)g_{1(-2)}(x,\varepsilon) = \mu^3[f_{1(-2)}'(x,\varepsilon) - f_{2(-2)}(x,\varepsilon)], \\ \varphi(x)\varphi'(x)g_{2(-2)}(x,\varepsilon) + f_{3(-2)}(x,\varepsilon) = 0, \\ \varphi(x)\varphi'(x)g_{3(-2)}(x,\varepsilon) + b(x)f_{1(-2)}(x,\varepsilon) + a(x)f_{2(-2)}(x,\varepsilon) = 0. \end{cases}$$

Since $\det \Phi(x) \equiv 0$, there exists a nontrivial solution to the system $\Phi(x) \cdot Z_{kr} = 0$, $r = \overline{-2,0}$ of the form

$$Z_{kr}(x) = colomn\Big(0, \frac{1}{\varphi'(x)}\, g_{2r}(x), \varphi\varphi'(x) g_{3r}(x), 0, g_{2r}(x), g_{3r}(x)\Big),$$

where $g_{kr}(x)$, $k = \overline{1,3}$, $r = \overline{-2,0}$, up to a certain moment, are arbitrary, sufficiently smooth functions with $x \in [-l; 0]$.

In this way, to obtain the solution of the system $\Phi(x) \cdot Z_{kr} = 0$, $r = \overline{-2,0}$, we move to the solutions of inhomogeneous systems $\Phi(x) \cdot Z_{kr}(x) = F \cdot Z_{k(r-3)}(x)$, $r \geq 1$. To do this, we consider differential equations of the form

$$2a(x) g'_{2(-2)}(x) + \big[-b(x) - \varphi'(x)(\varphi(x)\varphi'(x))'\big] g_{2(-2)}(x) = 0, \tag{6.6}$$

and

$$-2g'_{3(-2)}(x) + \Big[\frac{\varphi''(x)}{\varphi'(x)} - \frac{b(x)}{a(x)}\Big] g_{3(-2)}(x) = 0.$$

In equation (6.6), we introduce the notation

$$b_2(x) = -b(x) - \varphi'^3(x) - \varphi(x)\varphi'(x)\varphi''(x).$$

Remind that

$$\varphi(x) = \left(\frac{3}{2}\int\limits_0^x \sqrt{-a(x)}\, dx\right)^{\frac{2}{3}}, \quad \varphi'(x) = \left(\int\limits_0^x \sqrt{-a(x)}\, dx\right)^{-\frac{1}{3}} \cdot \sqrt{-a(x)}.$$

To find the components, we use the equations

$$g'_{2(-2)}(x) - \frac{1}{x}\Big[\frac{b_2(x)}{2\widetilde{a}(x)}\Big] g_{2(-2)}(x) = 0, \tag{6.7}$$

$$g'_{3(-2)}(x) - \frac{1}{x}\Big[\frac{b_3(x)}{2\widetilde{a}(x)}\Big] g_{3(-2)}(x) = 0. \tag{6.8}$$

From (6.8), we get

$$g'_{3(-2)}(x) - \frac{b_3(x)}{x\widetilde{a}(x)}\, g_{3(-2)}(x) = 0, \tag{6.9}$$

$$g_{3(-2)}(x) = \int\limits_0^x \frac{b_3(x)}{x\widetilde{a}(x)}\, g_{3(-2)}(x)\, dx = 0.$$

So, we write the solution for (6.9) in the form

$$g_{3(-2)}(x) = g^0_{3(-2)} \cdot \exp\Big\{\int \frac{b_3(x)}{x}\, dx\Big\}, \quad \overline{Z}^{part.}_{k1}(x) = colomn(\overline{z}_{k1}, \overline{z}_{k2}, \overline{z}_{k3}, \overline{z}_{k4}, \overline{z}_{k5}, \overline{z}_{k6}),$$

where

$$\overline{z}_{k1} = \frac{g^0_{2(-2)}(x) \cdot x^{\rho_2}}{\varphi'(x)}, \quad \overline{z}_{k2} = \frac{-g^0_{2(-2)}(x) \cdot x^{\rho_2} + g_{31}(x)}{\varphi'(x)},$$

$$\overline{z}_{k3} = \frac{-g^0_{2(-2)}(x) \cdot x^{\rho_2} + \frac{b(x) g^0_{3(-2)}(x) \cdot x^{\rho_3}}{a(x)} - a(x) g_{21}(x)}{\varphi'(x)}, \quad \overline{z}_{k4} = \frac{-g^0_{3(-2)}(x) \cdot x^{\rho_3}}{-a(x)},$$

$$\overline{z}_{k5} = g_{21}(x), \quad \overline{z}_{k6} = g_{31}(x),$$

where g_{k1}, $k = \overline{2,3}$, up to a certain moment, are arbitrary, sufficiently smooth functions with $x \in [-l; 0]$.

To ensure the construction of a uniform asymptotics of the solution of equation (2.1) on the entire segment of a relatively small parameter, it is necessary that the requirement $\rho \in N$ be fulfilled. Since $\frac{b(0)}{\widetilde{a}(0)} = \rho$ must be a natural number, we consider the following cases.

Case 1. Let $\rho = 2n$ be an even number, $n \in N$. Then, using the above notation, we obtain that $\rho_2 = n - \frac{1}{2}$ is not a natural number, but $\rho_3 = n$ is a natural number and $\rho_3 = 0$ when $\rho = 0$. We have

$$\frac{b_j(x)}{x\widetilde{a}(x)} = \frac{\rho}{x} + \widetilde{R}_j^{part.}(x),$$

where $\widetilde{R}_j^{part.}(x)$ is an analytic function in the vicinity of the turning point.

If $\rho_2 = n - \frac{1}{2}$ is a sufficiently large number, then to construct the asymptotics of a linearly independent solution of system (2.1) with a certain tonality of a relatively small parameter $\varepsilon > 0$, you can use solutions (6.2). Since in this case $\rho_3 = \frac{1}{2}\rho$ is a non-negative integer, using the general solution of equations of the form (6.8) and partial solutions of equations of the form (6.7), we construct the asymptotics of a linearly independent solution of system (2.1) with arbitrary accuracy of a relatively small parameter $\varepsilon > 0$ on the entire segment $[-l, 0]$, including the turning point.

Case 2. Let $\rho = 2n - 1$ be an odd number, $n \in N$. Then $\rho_2 = n - 1$ is a non-negative integer and $\rho_3 = n - \frac{1}{2}$ is not a natural number. In this case, to construct the asymptotics of a linearly independent solution of system (2.1) with a certain tonality of a relatively small parameter $\varepsilon > 0$, we use the general solution of equations of the form (6.7) and partial solutions of equations of the form (6.8).

We consider $\rho = 2n$, and $\rho_3 = n$, $n \in N$. Then, according to Case 1, the partial and sufficiently smooth solutions of equations (6.7) and (6.8) are $\omega_{10}(x) \equiv g_{2(-2)}(x) \equiv 0$,

$$g_{3(-2)}(x) = g^0_{3(-2)} \cdot \exp\left\{ \int \frac{b_3(x)}{2a(x)}\, dx \right\} \equiv g^0_{3(-2)} x^{\rho_2} \widetilde{g}(x),$$

when $r = 0$, where $\widetilde{g}(x)$ is a sufficiently smooth function for $x \in [-l, 0]$, provided that $\widetilde{g}(0) \neq 0$.

The solution of system (6.5) for $r = 0$ looks like

$$\overline{\omega}_{10}(x) = \overline{\omega}^0_{10} \cdot \exp\left\{ -\int \frac{b(x)}{a(x)}\, dx \right\} + \exp\left\{ -\int \frac{b(x)}{a(x)}\, dx \right\} \cdot \frac{h(x)}{a(x)} \exp\left\{ \int \frac{b(x)}{a(x)}\, dx \right\},$$

$$\overline{\omega}_{20}(x) = \overline{\omega}_{10}{}' \cdot \varphi'(x) \cdot \frac{g^0_{3(-2)} x^{\rho_3}}{a(x)}, \quad \overline{\omega}_{30}(x) = \varphi'(x) \cdot g^0_{2(-2)} x^{\rho_2}.$$

Continuing to solve the systems of iterative equations with (6.5), we find all the components $\omega_{kr}(x)$.

Theorem. *Let the system of singularly perturbed differential equations (2.1) satisfy the conditions:*

C 1. $A_0(x),\ H(x) \in C^\infty[-l, 0]$.

C 2. $a(x) = x\widetilde{a}(x)$, $\widetilde{a}(x) < 0$, $b(x) < 0$.

Then we can construct a formal solution

$$Y_k^{hom.}(x, \varepsilon^{-\frac{2}{3}}\varphi(x), \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \left[\alpha_{ikr}(x) U_i(\varepsilon^{-\frac{2}{3}}\varphi(x)) + \varepsilon^{\frac{1}{3}} \beta_{ikr}(x) U_i'(\varepsilon^{-\frac{2}{3}}\varphi(x)) \right]$$
$$+ \sum_{r=-2}^{\infty} \mu^r \left[f_{kr}(x)\nu(t) + \mu g_{kr}(x)\nu'(t) + \overline{\omega}_{kr}(x) \right],$$

$\alpha_{ikr}(x)$ and $\beta_{ikr}(x)$ are determined from (5.1), and $f_{kr}(x)$, $g_{kr}(x)$, $\omega_{kr}(x)$ from (6.1).

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