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THE STRUCTURE OF MORSE AND CODIMENSION ONE GRADIENT FLOWS ON THE SPHERE WITH HOLES

Abstract. We describe all possible topological structures of Morse flows and codimension one gradient flows generated by vector fields emerging in typical one-parameter bifurcations. Research is held on a 2-sphere with holes in the case where the number of singular points of flows is at most six.

For that purpose, we construct topological invariants of these flows which are actually the graphs endowed with certain information. In this case, they are separatrix skeletons and distinguished graphs of flows. For example, the saddle-node singularity is specified by selecting a separatrix in the skeleton of the flow before the bifurcation, whereas the saddle connection is specified by a separatrix which connects two saddles. Apart from that, we construct special codes for these graphs so that we can regain a skeleton and a graph from the code and thus recover the whole flow on a certain surface.

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რეზიუმე. აღწერილია ყველა შესაძლო ტოპოლოგიური სტრუქტურა მორსის ნაკადებისა და ერთის ტოლი კოგანზომილების მქონე გრადიენტური ნაკადებისთვის, რომლებიც გენერირდება ტიპურ ერთპარამეტრიან ბიფურკაციებში წარმოქმნილი ვექტორული ველებით. კვლევა ტარდება 2-სფეროზე ხვრელებით იმ შემთხვევაში, როდესაც ნაკადების სინგულარული წერტილების რაოდენობა მაქსიმუმ ექვსის ტოლია.

ამ მიზნით, აგებულია ამ ნაკადების ტოპოლოგიური ინვარიანტები, რომლებიც ფაქტიურად წარმოადგენს გარკვეული ინფორმაციით აღჭურვილ გრაფებს. მოცემულ შემთხვევაში ისინი წარმოადგენს სეპარატრიქსულ ჩონჩხებსა და ნაკადების გამორჩეულ გრაფებს. მაგალითად, უნაგირ-კვანძის სინგულარობა განისაზღვრება ნაკადის ჩონჩხში სეპარატრიქსის არჩევით ბიფურკაციამდე, ხოლო უნაგირთა კავშირი განისაზღვრება სეპარატრიქსით, რომელიც აკავშირებს ორ უნაგირს. გარდა ამისა, აგებულია სპეციალური კოდები ამ გრაფებისთვის, რათა კოდიდან აღვადგინოთ ჩონჩხი და გრაფიკი და ამით აღვადგინოთ მთელი ნაკადი გარკვეულ ზედაპირზე.

1 Introduction

In 1970, Palis and Smale [10] established that Morse–Smale gradient vector fields are structurally stable within the set of all gradient fields $\mathrm{Grad}(M)$ with the C^k topology ($k \geq 3$) on a smooth manifold M. Additionally, codimension one gradient vector fields are those that occur in typical 1-parameter families of gradient vector fields and are not Morse–Smale vector fields. In 1983, Palis and Takens [11] demonstrated that these fields can be categorized into two types: SN-fields, which have a single saddle-node singularity, and SC-fields, which include a saddle connection. Thus, in each case, only one condition from the definition of a Morse–Smale vector field is violated. Specifically, they are structurally stable within the set $\mathrm{Grad}(M) \setminus \mathrm{MS}(M)$, and are also an open and dense subset in this space. In case such vector fields generate a flow, the respective flows are called *codimension one flows*.

A smooth vector field defines a flow only if it is tangent to the boundary. A flow is considered gradient, Morse–Smale, or codimension one flow if the generating vector fields possess the respective properties. The description of Morse–Smale and codimension one flows on manifolds with boundary can be found in [7] and [5]. The structural stability of these flows within the relevant spaces, as well as the openness and overall density of their sets in these spaces, have also been established.

For topological classification of flows, complete topological invariants can be established. These invariants often take the form of graphs that possess specific properties and are endowed with additional information. Furthermore, two flows are considered topologically equivalent if there exists an isomorphism between their graphs that preserves this information. The effectiveness of these invariants is determined by the ability to identify all structures of dynamical systems with a specific number of singular points. For instance, a separatrix skeleton serves as a complete topological invariant for Morse–Smale flows [8, 9, 13].

We can refine the invariant for the Morse flows (which are Morse–Smale flows lacking closed trajectories) that are topologically equivalent to flows generated by gradient vector fields of Morse functions. This invariant is a graph embedded in a surface, where the vertices represent sources and the edges correspond to the stable manifolds of saddles. The topological structure of Morse flows can also be studied using Lyapunov functions, which are the Morse functions. Their classification was obtained in [2, 4, 20].

A rotation system [3] is frequently used to define an embedding of a graph into a surface. This serves as a Peixoto invariant [13] for the Morse flows. When dealing with a flow that has a single source, the chord diagrams are a convenient means of representation [6]. To create a chord diagram, codes are introduced. The codes consist of the numbers representing the chords whose endpoints connect while moving around the circle. However, the potential for ambiguity in numbering the chords or selecting a starting point can make it challenging to compare the resulting invariants.

Additionally, complete topological invariants were developed for flows with gradient dynamics in dimension 2 [17] and dimension 3 [14, 19]. In [15, 18], flows with collective dynamics on the sphere were studied, which, in addition to Hamiltonian regions, are the Morse flows on the sphere with holes. In this case, the saddle points on the boundary of the gradient region contain three hyperbolic sectors, unlike the flows we are considering, which contain two hyperbolic sectors for such points.

The purpose of this paper is to construct a complete invariant for the Morse and gradient codimension one flow on the sphere with holes which resembles a chord diagram for Morse flows. This invariant has a marked point in the diagram, which allows us to define uniquely a number code of the flow. The invariants we constructed are generalizations of the distinguishing graph of Peixoto and the Oshemkov–Sharko code, which were developed for Morse flows on closed surfaces.

In Section 2, the basic definitions concerning gradient vector fields and flows on compact surfaces are given; in particular, one can find there precise definitions of Morse flows and all the codimension one gradient flows on compact surfaces.

In Section 3, we give a definition of distinguishing graph for Morse flows and some of codimension one gradient flows and present the theorems confirming that these graphs are their complete topological invariants.

In Section 4, we provide the so-called flow codes which are constructed for Morse codes according to their distinguishing graphs, and give theorems on the isomorphism of these graphs depending on similarity of the codes, as well as theorems on the realization of distinguishing graphs through flow codes. Section 5 gives the same results for codimension one gradient flows.

2 Preliminaries

2.1 Smooth flows and vector fields on compact surfaces

In this paper, specific types of smooth gradient flows and vector fields on a sphere S^2 with n holes, $n \geq 1$, are considered. More precisely, suppose that we have a centre of stereographic projection $\xi : S^2 \to \mathbb{R}^2$ inside one of n holes. Then ξ is an embedding and $M := \xi(S^2)$ is a compact connected subset of the Euclidean plane with a finite number of connected components of its boundary. We also have a new C^∞-vector field V on M, derived from the appropriate one on S^2, which can be expressed through the differential equations

$$\frac{dx}{dt} = V(x), \quad x \in \mathbb{R}^2.$$

In accordance with the Picard theorem, this equation has a unique solution

$$X_{x_0}(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

for a fixed arbitrary initial condition $x(0) = x_0 \in \mathbb{R}^2$ (Cauchy problem) which can be depicted as an *oriented trajectory* on M. We consider a mapping $g : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, $g(x,t) = X_x(t)$, $x \in \mathbb{R}^2$, $t \in \mathbb{R}$.

Let M be a smooth compact surface (2-manifold) in this subsection.

Definition 2.1 ([12])**.** A smooth mapping $g : M \times \mathbb{R} \to M$ is called a *smooth dynamical system*, or a *flow*, on M if:

(1) $g(x, 0) = x$ for each $x \in M$;

(2) $g(x, t+s) = g(g(x,s), t)$ for all $t, s \in \mathbb{R}$, $x \in M$.

Henceforth, we denote the set of sources of a flow g on M as $\Omega_0(g)$, the set of saddles of a flow g on M as $\Omega_1(g)$ and the set of sinks of a flow g on M as $\Omega_2(g)$. The set of all singular points of a flow g on M is denoted as $\mathrm{Sign}(g)$.

Definition 2.2. Suppose that M_1 and M_2 are smooth compact surfaces. Two flows $g_i : M_i \times \mathbb{R} \to M$, $i = 1, 2$, are *topologically* or *trajectory equivalent* if there exists a homeomorphism $h : M_1 \to M_2$ which maps each trajectory of the first flow onto a trajectory of the second one without reversing orientations.

Suppose that $X : M \to TM$ is a gradient C^k-vector field on M ($k \geq 3$) of a smooth function $f : M \to \mathbb{R}$ for some Riemannian metric on M, which is fixed by us, and $g : M \times \mathbb{R} \to M$ is a flow on M generated by the vector field X. Then X must be tangent to ∂M if $\partial M \neq \varnothing$, otherwise it does not generate a flow and the contradiction arises.

Definition 2.3. A set

$$W^s(p) = \left\{ x \in M : \lim_{t \to +\infty} g(x,t) = p \right\} \quad \left(W^u(p) = \left\{ x \in M : \lim_{t \to -\infty} g(x,t) = p \right\} \right)$$

is called a *stable (unstable) manifold* of a singular point p of a flow g.

In [16], one can become familiar with the notions of a hyperbolic sector and a separatrix.

Definition 2.4. A separatrix γ of a flow g on M is called *unstable* if $\alpha(\gamma) \in \Omega_1(g)$ and $\omega(\gamma) \in \Omega_2(g)$. Besides, a separatrix γ is called *stable* if $\alpha(\gamma) \in \Omega_0(g)$ and $\omega(\gamma) \in \Omega_1(g)$.

Definition 2.5. A non-closed trajectory γ is called a *separatrix connection*, or a *saddle connection*, if $\alpha(\gamma) \cup \omega(\gamma) \subset \Omega_1(g)$.

Definition 2.6. A *separatrix skeleton* of a gradient flow g (vector field X) on a surface M is the surface M on which singular points, boundary trajectories, separatrices and defined orientation of these trajectories are highlighted. In other words, a separatrix skeleton of a flow g on a surface M is a directed graph (g) embedded in the surface M. The vertices of the graph (g) are the singular points, and its edges are the separatrices and trajectories belonging to ∂M. The orientations of the edges of (g) are determined by the direction of movement along the corresponding trajectories.

For convenience, in a separatrix skeleton we colour unstable internal separatrices in green and stable internal ones in red, whereas all the boundary trajectories and saddle connections are coloured in black.

Definition 2.7. A singular point p of a vector field $X = (X_1(x_1, x_2), X_2(x_1, x_2))$ given in some local coordinates (x_1, x_2) in a neighbourhood U of p on a compact surface M is called *non-degenerated* or *hyperbolic* if each eigenvalue of the Jacobi matrix of X in the same local coordinates in p has a non-zero real part. Otherwise, such a singular point is called *degenerated* or *non-hyperbolic*.

Let g be a gradient flow on M generated by a smooth vector field X. A singular point p of g is called *non-degenerated* or *hyperbolic* if it is so for the vector field X. Otherwise, such a singular point is called *degenerated* or *non-hyperbolic*.

If a non-degenerated singular point p of a flow g (a vector field X) on a compact surface M lies on its interior, then $p \in \Omega_0(g) \cup \Omega_1(g) \cup \Omega_2(g)$.

Definition 2.8. Suppose that M_1 and M_2 are smooth compact surfaces. Two flows $g_i : M_i \times \mathbb{R} \to M$, $i = 1, 2$, generated respectively by the vector fields X_i, $i = 1, 2$, are *strongly topologically equivalent* if they are topologically equivalent and the appropriate homeomorphism $h : M_1 \to M_2$ does not make degenerate points for X_1 non-degenerate for X_2, and vice versa.

Definition 2.9. A vector field X on a compact surface M is called a *Morse vector field* if:

(1) it has a finite number of singular points, and all the singular points are non-degenerate;

(2) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, then $T_x W^s(u_1) + T_x W^u(u_2) = T_x M$;

(3) α-limit and ω-limit sets of each trajectory of g are singular points;

(4) all the tangent vectors of X defined on ∂M are tangent to it.

If a Morse vector field generates a flow on M, such a flow will be called a *Morse flow*.

The separatrix skeleton is a complete topological invariant of a Morse flow on a compact surface.

However, this invariant can be simplified by removing internal saddles and green (unstable) separatrices. The remaining singular points are 0-*cells*, the rest of the boundary trajectories and red (stable) separatrices are 1-*cells*, and the components of the complement to the 0- and 1-cells are 2-*cells*.

Each 2-cell has a unique sink: either one of the singular points on the boundary or an interior point. Since, up to homeomorphism, saddle points on the boundary of a cell can be connected by curves without intersection with the sink uniquely, the cell decomposition we described carries the same information about the flow structure as the separatrix skeleton.

2.2 Codimension one gradient flows on compact surfaces

Suppose that we have a smooth compact surface (2-manifold) M. All possible typical bifurcations in one-parameter families of gradient flows on compact surfaces are described in [5, Theorem I]. They will be briefly described in this subsection below.

We examine the bifurcations in the neighbourhood U of the singular point ($x = 0$, $y = 0$) as a family of vector fields $V = V(x, y, a)$ depending on the parameter $a \in [-1, 1]$ with $a = 0$ being the bifurcation point. In all cases, the vector field $V(x, y, 0)$ can be obtained from $V(x, y, -1)$ by collapsing to a trajectory that connects the singular points. Thus, at $a = 0$, we have a vector field which generates a flow.

2.2.1 Internal saddle-node (SN)

There are two types of internal saddle-nodes that arise at $a = 0$ for such families of vector fields:

(1) $V(x, y, a) = \{x, y^2 + a\}$ if the node is a source (SN_+ bifurcation, see Fig. 1);

(2) $V(x, y, a) = \{-x, -y^2 + a\}$ if the node is a sink (SN_- bifurcation).

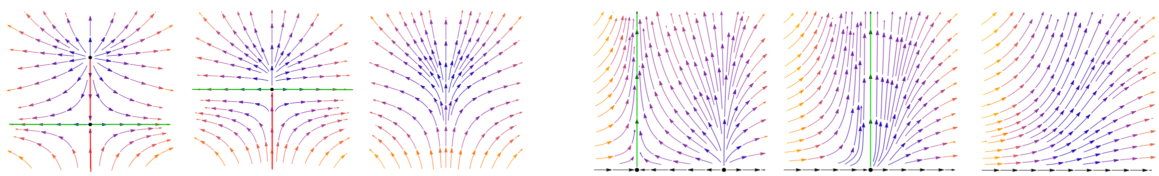
Here, x, y are the local coordinates, a is a real parameter.

In Fig. 1 (a), the vector field is shown before the bifurcation ($a = -1$), in Fig. 1 (b), the vector field at the moment of bifurcation ($a = 0$), and in Fig. 1 (c), the vector field after the bifurcation ($a = 1$).

This bifurcation corresponds to the appropriate one at $a = 0$ in the following family of functions:

$$f(x, y, a) = \pm\frac{1}{2}x^2 + \frac{1}{3}y^3 + ay,$$

where x, y are the local coordinates, a is a real parameter.



(a) (b) (c) (d) (e) (f)

Figure 1: SN_+ bifurcation (a)–(c) and BSN_+ bifurcation (d)–(f)

Definition 2.10. A vector field X on a compact surface M generating a flow g on M is called an SN_+-*vector field* (SN_--*vector field*) if:

(1) it has a finite number of singular points, and it has only one singular non-hyperbolic point sn on the interior of M, so sn is actually of saddle-node type;

(2) there is a closed neighbourhood U of sn and a homeomorphism $h : U \to S$, where

$$S = \{(x, y) \in \mathbb{R}^2 : \ -1 \leq x \leq 1, \ -1 \leq y \leq 1\},$$

which maps the intersections of trajectories of the flow g on M with U into the intersections of trajectories of the flow g_1 on $\mathbb{R}^2$, generated by the gradient vector field

$$V(x, y) = \{x, y^2\} \quad (V(x, y) = \{-x, -y^2\}), \quad x, y \in \mathbb{R}^2,$$

with S (see Fig. 1 (b));

(3) α-limit and ω-limit sets of each trajectory of g are singular points;

(4) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, then $T_xW^s(u_1) + T_xW^u(u_2) = T_xM$, and if for some separatrix $\gamma \subset \mathrm{Int}(M)$, the equality

$$\alpha(\gamma) = sn \quad (\omega(\gamma) = sn)$$

is true, then $\omega(\gamma)$ is a sink ($\alpha(\gamma)$ is a source);

(5) all the tangent vectors of X defined on ∂M are tangent to it.

Definition 2.11. A flow g on a compact surface M is called an SN_+-*flow* (SN_--*flow*) if it is a flow on M generated by an SN_+-vector field (SN_--vector field).

2.2.2 Boundary saddle-node (BSN)

There are two types of boundary saddle-nodes that arise at $a = 0$ for the following families of vector fields:

(1) $V_1(x, y, a) = \{x^2 + a, y\}$, $y \geq 0$, if the node is a source (BSN_+ bifurcation, see Fig. 1);

(2) $V_2(x, y, a) = \{-x^2 + a, -y\}$, $y \geq 0$, if the node is a sink (BSN_- bifurcation).

Here, x, y are the local coordinates, a is a real parameter.

In Fig. 1 (d), the vector field is shown before the bifurcation ($a = -1$), in Fig. 1 (e), the vector field at the moment of bifurcation ($a = 0$), and in Fig. 1 (f), the vector field after the bifurcation ($a = 1$).

This bifurcation corresponds to the appropriate one at $a = 0$ in the following family of functions:

$$f(x, y, a) = \pm\frac{1}{3}\,x^3 + ax \pm \frac{1}{2}\,y^2,$$

where x, y are the local coordinates, a is a real parameter.

If at any moment we cut the phase diagram of the SN-flow whose vector field is defined by the equation

$$V(x, y) = \big\{ \pm x,\ \pm y^2 \big\}, \quad x, y \in \mathbb{R}^2,$$

along the axis of symmetry ($x = 0$), then a BSN-flow will occur in each part.

Definition 2.12. A vector field X on a compact surface M with a boundary ∂M generating a flow g on M is called a BSN_+-*vector field* (BSN_--*vector field*) if:

(1) it has a finite number of singular points, and it has only one singular non-hyperbolic point bsn on the boundary of M, so bsn is actually of saddle–node type;

(2) there is a closed neighbourhood U of bsn and a homeomorphism $h : U \to S$, where $S = \{(x, y) \in \mathbb{R}^2 :\ -1 \leq x \leq 1,\ 0 \leq y \leq 1\}$, which maps the intersections of trajectories of the flow g on M with U into the intersections of trajectories of the flow g_1 on $\mathbb{R}^2$, generated by the gradient vector field

$$V(x, y) = \big\{x^2, y\big\} \ \ (V(x, y) = \{-x^2, -y\}), \quad x, y \in \mathbb{R}^2,$$

with S (see Fig. 1 (e));

(3) α-limit and ω-limit sets of each trajectory of g are singular points;

(4) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, then $T_x W^s(u_1) + T_x W^u(u_2) = T_x M$, and if for some separatrix $\gamma \subset \mathrm{Int}(M)$, the equality

$$\alpha(\gamma) = sn \ \ (\omega(\gamma) = sn)$$

is true, then $\omega(\gamma)$ is a sink ($\alpha(\gamma)$ is a source);

(5) all the tangent vectors of X defined on ∂M are tangent to it.

Definition 2.13. A flow g on a compact surface M with a boundary ∂M is called a BSN_+-*flow* (BSN_--*flow*) if it is a flow on M generated by a BSN_+-vector field (BSN_--vector field).

2.2.3 Degenerated saddle on the boundary (HS)

There are two types of degenerated saddles on the boundary that arise at $a = 0$ for the following families of vector fields:

(1) $V_2(x, y, a) = \{x, -y^2 - ay\}$, $y \geq 0$ (HS_+ bifurcation, see Fig. 2);

(2) $V_1(x, y, a) = \{-x, y^2 + ay\}$, $y \geq 0$ (HS_- bifurcation);

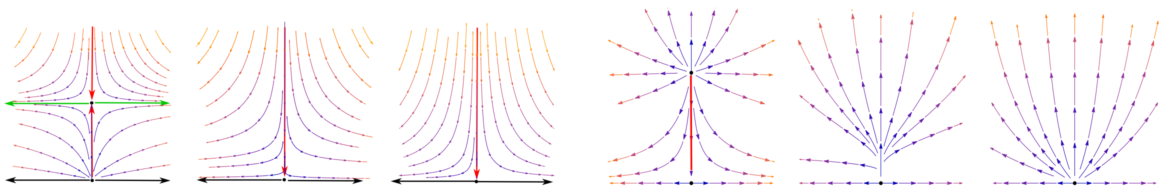
Here, x, y are the local coordinates, a is a real parameter.

In Fig. 2 (a), the vector field is shown before the bifurcation ($a = -1$), in Fig. 2 (b), the vector field at the moment of bifurcation ($a = 0$), and in Fig. 2 (c), the vector field after the bifurcation ($a = 1$).

This bifurcation corresponds to the appropriate one at $a = 0$ in the following family of functions:

$$f(x, y, a) = \mp\frac{1}{3}\,y^3 + \mp ay \pm \frac{1}{2}\,x^2,$$

where x, y are the local coordinates, a is a real parameter.



(a) (b) (c) (d) (e) (f)

Figure 2: HS_+ bifurcation (a)–(c) and HN_+ bifurcation (d)–(f)

Definition 2.14. A vector field X on a compact surface M with a boundary ∂M generating a flow g on M is an HS_+-*vector field* (HS_--*vector field*) if:

(1) it has a finite number of singular points, and it has only one degenerated singular point, degenerated saddle hs, belonging to the boundary ∂M;

(2) there is a closed neighbourhood U of hs and a homeomorphism $h : U \to S$, where

$$S = \{(x, y) \in \mathbb{R}^2 : \ -1 \le x \le 1, \ \ 0 \le y \le 1\},$$

which maps the intersections of trajectories of the flow g on M with U into the intersections of trajectories of the flow g_1 on $\mathbb{R}^2$, generated by the gradient vector field

$$V(x, y) = \{x, -y^2\} \ \ (V(x, y) = \{-x, y^2\}), \ \ x, y \in \mathbb{R}^2,$$

with S (see Fig. 2 (b));

(3) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, then $T_xW^s(u_1) + T_xW^u(u_2) = T_xM$;

(4) α-limit and ω-limit sets of each trajectory of g are singular points;

(5) all the tangent vectors of X defined on ∂M are tangent to it.

Definition 2.15. A flow g on a compact surface M with a boundary ∂M is called an HS_+-*flow* (HS_--*flow*) if it is a flow on M generated by an HS_+-vector field (HS_--vector field).

Remark 2.1. Note that any HS-flow on M is topologically equivalent to some Morse flow on M.

2.2.4 Degenerated node on the boundary (HN)

There are two types of degenerated nodes on the boundary that arise at $a = 0$ for the following families of vector fields:

(1) $V_1(x, y, a) = \{x, y^2 + ay\}$, $y \ge 0$, if the node is a source (HN_+ bifurcation, see Fig. 2);

(2) $V_2(x, y, a) = \{-x, -y^2 - ay\}$, $y \ge 0$, if the node is a sink (HN_- bifurcation).

Here, x, y are the local coordinates, a is a real parameter.

In Fig. 2 (d), the vector field is shown before the bifurcation ($a = -1$), in Fig. 2 (e), the vector field at the moment of bifurcation ($a = 0$), and in Fig. 2 (f), the vector field after the bifurcation ($a = 1$).

This bifurcation corresponds to the appropriate one at $a = 0$ in the following family of functions:

$$f(x, y, a) = \pm\frac{1}{2}\,x^2 \pm \frac{1}{3}\,y^3 \pm \frac{1}{2}\,ay^2,$$

where x, y are the local coordinates, a is a real parameter.

If at any moment we cut the phase diagram of the SN-flow, whose vector field is defined by equation

$$V(x, y) = \{\pm x, \pm y^2\}, \quad x, y \in \mathbb{R}^2,$$

along the axis of symmetry ($y = 0$), then an HN-flow will occur in an upper half of the Euclidean plane, whereas an HS-flow will occur in its lower part.

Before cutting, we had a flow with a degenerated singular point $(0, 0)$ of a saddle-node type, but after cutting, we got instead two degenerated points: a degenerated saddle on the boundary and a degenerated node on the boundary.

Definition 2.16. A vector field X on a compact surface M with a boundary ∂M generating a flow g on M is an HN_+-*vector field* (HN_--*vector field*) if:

(1) it has a finite number of singular points and only one degenerated singular point, degenerated source (sink) hn, belonging to the boundary ∂M;

(2) there is a closed neighbourhood U of hs and a homeomorphism $h : U \to S$, where $S = \{(x, y) \in \mathbb{R}^2 :\ -1 \le x \le 1,\ 0 \le y \le 1\}$, which maps the intersections of trajectories of the flow g on M with U into the intersections of trajectories of the flow g_1 on $\mathbb{R}^2$, generated by the gradient vector field

$$V(x, y) = \{x, y^2\} \quad (V(x, y) = \{-x, -y^2\}), \quad x, y \in \mathbb{R}^2,$$

with S (see Fig. 2 (e));

(3) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, then $T_xW^s(u_1) + T_xW^u(u_2) = T_xM$;

(4) α-limit and ω-limit sets of each trajectory of g are singular points;

(5) all the tangent vectors of X defined on ∂M are tangent to it.

Definition 2.17. A flow g on a compact surface M with a boundary ∂M is called an HN_+-*flow* (HN_--*flow*) if it is a flow on M generated by an HN_+-vector field (HN_--vector field).

Remark 2.2. Note that any HN-flow on M is topologically equivalent to some Morse flow on M.

2.2.5 Double saddle on the boundary (BDS)

BDS bifurcation (see Fig. 3) arises at $a = 0$ for the following family of vector fields:

$$V(x, y, a) = \{3x^2 - 3y^2 + a, -6xy\}, \quad y \ge 0.$$

Here, x, y are the local coordinates, a is a real parameter.

In Fig. 3 (a), the vector field is shown before the bifurcation ($a = -1$), in Fig. 3 (b), the vector field at the moment of bifurcation ($a = 0$), and in Fig. 3 (c), the vector field after the bifurcation ($a = 1$).

This bifurcation corresponds to the appropriate one at $a = 0$ in the following family of functions:

$$f(x, y, a) = \mathrm{Re}(x + \mathrm{i}\,y)^3 + ax = x^3 - 3xy^2 + ax,$$

where x, y are the local coordinates, a is a real parameter.

Definition 2.18. A vector field X on a compact surface M with a boundary ∂M generating a flow g on M is a BDS-*vector field* if:

(1) it has a finite number of singular points and only one degenerated singular point, double saddle bds, belonging to the boundary ∂M;

(2) there is a closed neighbourhood U of bsn and a homeomorphism $h : U \to S$, where $S = \{(x, y) \in \mathbb{R}^2 : \ -1 \le x \le 1,\ 0 \le y \le 1\}$, which maps the intersections of trajectories of the flow g on M with U into the intersections of trajectories of the flow g_1 on $\mathbb{R}^2$, generated by the gradient vector field

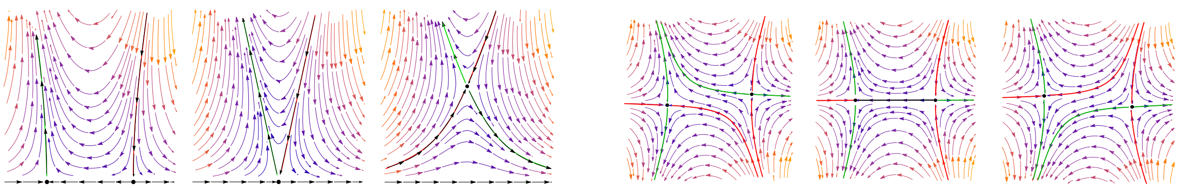
$$V(x, y) = \{3x^2 - 3y^2, -6xy\}, \quad x, y \in \mathbb{R}^2,$$

with S (see Fig. 3 (b));

(3) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, then $T_xW^s(u_1) + T_xW^u(u_2) = T_xM$;

(4) α-limit and ω-limit sets of each trajectory of g are singular points;

(5) all the tangent vectors of X defined on ∂M are tangent to it.



(a) (b) (c) (d) (e) (f)

Figure 3: BDS bifurcation (a)–(c) and SC bifurcation (d)–(f)

Definition 2.19. A flow g on a compact surface M with a boundary ∂M is called a BDS-*flow* if it is a flow on M generated by a BDS-vector field.

2.2.6 Internal saddle connection (SC)

SC bifurcation (see Fig. 3) arises at $a = 0$ for the following family of vector fields:

$$V(x, y, a) = \big\{x^2 - y^2 - 1, -2xy + a\big\}.$$

Here, x, y are the local coordinates, a is a real parameter.

In Fig. 3 (d) the vector field is shown before the bifurcation ($a = -1$), in Fig. 3 (e), the vector field at the moment of bifurcation ($a = 0$), and in Fig. 3 (f), the vector field after the bifurcation ($a = 1$).

Let us describe one of the possible situations when such a bifurcation emerges. We consider the torus T^2 in the Euclidean space $\mathbb{R}^3$ which can be parametrized as

$$r(u, v) = \big\{(0.5\cos u + 2)\sin v,\ 0.5\sin u,\ (0.5\cos u + 2)\cos v\big\}, \quad u \in [0, 2\pi], \quad v \in [0, 2\pi].$$

Suppose that we have an ax $Oz'(a)$ which is collinear to the vector $\vec{v} := (0, \sin a, \cos a)$ for each $a \in [-\pi/4, \pi/4]$. One can put a set of real numbers on it. More precisely, if there is a distance d between the point x on $Oz'(a)$, then we put d near it if $\vec{Ox}$ and $\vec{v}$ are collinear, and we put $-d$ near it, otherwise. In this case, $Oz'(0) = Oz$. Then, for each $a \in [-\pi/4, \pi/4]$, we define a height function $f(x, y, z, a)$ "respectively to the ax $Oz'(a)$". It can be defined as

$$f(x, y, z, a) = \sin a \cdot y + \cos a \cdot z, \quad (x, y, z) \in \mathbb{R}^2, \quad a \in \Big[-\frac{\pi}{4}, \frac{\pi}{4}\Big].$$

The family f of such functions can be given on T^2 in the local coordinates (u, v):

$$f(u, v, a) = 0.5\sin a \cdot \sin u + \cos a \cdot \cos v \cdot (0.5\cos u + 2), \quad u \in [0, 2\pi], \quad v \in [0, 2\pi].$$

It generates the next family of vector fields on T^2:

$$V'(u, v, a) = \big\{0.5\sin a\cos u - 0.5\cos a\sin u\cos v, -\cos a\sin v(0.5\cos u + 2)\big\}, \quad u \in [0, 2\pi], \quad v \in [0, 2\pi].$$

Now let us consider a subsurface with boundaries on T^2, where $v \in [\pi/2 - 0.025, 3\pi/2 + 0.025]$, and glue to its boundaries two hemispheres of diameter 1, considering the same family of functions on it. We got new surface L and new family of vector fields V'' generated by f. The vector field for $a = 0$ on L has only one saddle connection γ unlike the one on a complete torus. Let us take a small enough closed neighbourhood W of γ which is homeomorphic to some rectangular closed subset of $\mathbb{R}^2$.

The topological behaviour of the obtained gradient vector fields in W is the same as the behaviour of ones depicted in Fig. 3, so the SC bifurcation corresponds to the appropriate one at $a = 0$ in the constructed family of functions.

Definition 2.20. A vector field X on a compact surface M generating a flow g on M is an SC-*vector field* if:

(1) it has a finite number of singular points, and all the singular points are non-degenerated;

(2) it has only one saddle connection γ belonging to $\mathrm{Int}(M)$, $\alpha(\gamma) \cup \omega(\gamma) \subset \mathrm{Int}(M)$;

(3) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, x does not belong to the saddle connection γ, then $T_xW^s(u_1) + T_xW^u(u_2) = T_xM$;

(4) α-limit and ω-limit sets of each trajectory of g are singular points;

(5) all the tangent vectors of X defined on ∂M are tangent to it.

Definition 2.21. A flow g on a compact surface M is called an SC-*flow* if it is a flow on M generated by an SC-vector field.

2.2.7 Half-boundary saddle connection (HSC)

The HSC bifurcation on a compact surface M with a boundary ∂M can be described by the same formula as the SC bifurcation, with a restriction on the area, where it is considered. This area is formed by cutting around the separatrix connection γ along a stable or unstable manifold of one of the saddles that does not contain γ.

If $\omega(\gamma) \in \partial M$, we have a bifurcation HSC_+, and if $\alpha(\gamma) \in \partial M$, then we have an HSC_- bifurcation. We obtain HSC_+ if we cut the bifurcation in Fig. 3 along the unstable manifold of the left saddle (and take the right area), while HSC_- is obtained from the left area of the stable manifold of the right saddle of the bifurcation in Fig. 3.

Definition 2.22. A vector field X on a compact surface M with a boundary ∂M generating a flow g on M is an HSC_+-*vector field* (HSC_--*vector field*) if:

(1) it has a finite number of singular points, and all the singular points are non-degenerated;

(2) it has only one saddle connection γ belonging to $\mathrm{Int}(M)$, $\alpha(\gamma) \in \mathrm{Int}(M)$ and $\omega(\gamma) \in \partial M$ ($\alpha(\gamma) \in \partial M$ and $\omega(\gamma) \in \mathrm{Int}(M)$);

(3) if $x \in \mathrm{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \mathrm{Sign}(g)$, x does not belong to the saddle connection γ, then $T_xW^s(u_1) + T_xW^u(u_2) = T_xM$;

(4) α-limit and ω-limit sets of each trajectory of g are singular points;

(5) all the tangent vectors of X defined on ∂M are tangent to it.

Definition 2.23. A flow g on a compact surface M with a boundary ∂M is called an HSC_+-*flow* (HSC_--*flow*) if it is a flow on M generated by an HSC_+-vector field (HSC_--vector field).

2.2.8 Boundary saddle connection (BSC)

The BSC bifurcation can be described by the same formula as SC, with a restriction on the area, where it is considered. This area is formed by cutting around the separatrix connection γ along the stable manifold of one saddle and the unstable manifold of another saddle, so that these manifolds do not contain γ. We obtain BSC if we cut the bifurcation in Fig. 3 along the unstable manifold of the left saddle and the stable manifold of the right saddle, taking the central part.

Definition 2.24. A vector field X on a compact surface M with a boundary ∂M generating a flow g on M is a BSC-*vector field* if:

(1) it has a finite number of singular points, and all the singular points are non-degenerated;

(2) it has only one saddle connection γ belonging to $\operatorname{Int}(M)$, $\alpha(\gamma) \cup \omega(\gamma) \subset \partial M$;

(3) if $x \in \operatorname{Int}(M) \cap W^s(u_1) \cap W^u(u_2)$, $u_1, u_2 \in \operatorname{Sign}(g)$, x does not belong to the saddle connection γ, then $T_x W^s(u_1) + T_x W^u(u_2) = T_x M$;

(4) α-limit and ω-limit sets of each trajectory of g are singular points;

(5) all the tangent vectors of X defined on ∂M are tangent to it.

Definition 2.25. A flow g on a compact surface M with a boundary ∂M is called a BSC-*flow* if it is a flow on M generated by a BSC-vector field.

According to [5], all the listed above flows are gradient; but it can also be proved in the same way as it was implied from [1, Lemma 1.6] that all the Morse flows on a compact surface are gradient. We will call such flows on compact surfaces *codimension one gradient flows*.

Our aim is to study all codimension one gradient flows and the Morse flows on some spheres with a finite number of holes by using specific flow codes defining in some cases their complete invariants, the distinguishing graphs.

3 Notion of distinguishing graph

Suppose that we have a flow g on a compact connected surface M being either a Morse flow, or an SC-flow, or a BSC-flow, or an HSC_+-flow, or an HSC_--flow. Let also γ be a saddle connection of g belonging to $\operatorname{Int}(M)$ if it has any. We are about to construct a distinguishing graph for these flows.

Definition 3.1. A connected neighbourhood U of a set $S \subset M$ is called *regular* if $(\operatorname{Sign}(g) \cap \overline{U}) \subset S$ and ∂U intersects trajectories of g at most once; moreover, if $S \cap \partial M = \varnothing$ then $\partial U \cap \partial M = \varnothing$.

For an undirected graph G, denote a set of its vertices as $V(G)$ and a set of its edges as $E(G)$. Also, denote

$$W := \bigcup_{p \in \Omega_1(g)} W^s(p), \quad \Omega_a(g) := \big\{p \in \Omega_1(g) : \ W^u(p) \subset \partial M\big\}.$$

We fix a union of sets with non-intersecting boundaries

$$U := \bigcup_{p \in \Omega_0(g)} U(p)$$

from the surface M, where $U(p)$ means a regular neighbourhood of a point p. A set $\partial(M \setminus U) \cup (W \cap (M \setminus U))$ will be defined as an undirected graph Γ, for which

$$V(\Gamma) = (\partial M \cap \partial U) \cup \big((\Omega_a(g) \cup \Omega_2(g)) \cap \partial M\big) \cup (W \cap \partial U) \cup \alpha(\gamma),$$

and $E(\Gamma)$ is a set of connected components of Γ separated by the elements from $V(\Gamma)$.

Definition 3.2. The graph Γ constructed above is called a *distinguishing graph* of the flow g if:

(1) its edges lying in ∂U have a second colour (they are dashed), its edges lying in $\mathrm{Int}(M \setminus U)$ have a first (thin black) colour, the other edges have a third (thick black) colour;

(2) the sinks of g have a white colour and the other vertices of Γ have a black colour.

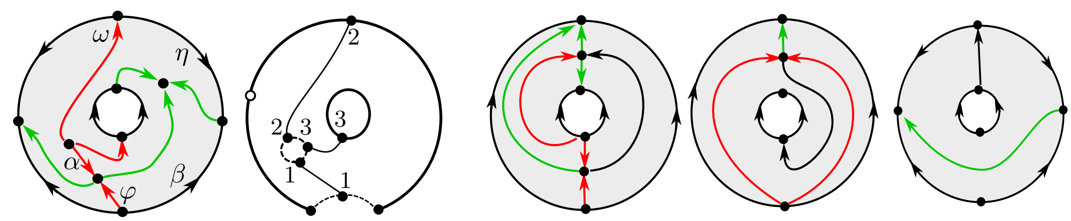
Denote all thin black edges in Γ as B and all their endpoints as T.

Definition 3.3. A *boundary cycle* of the graph Γ is a connected component of $\Gamma \setminus B$.

The orientation of the surface defines the orientation of each boundary cycle. Since a distinguishing graph is connected, there exists an edge of the first color that connects any boundary cycle to others.

By a *T-graph* we will mean a connected tree with three edges where one vertex has a degree 3. Obviously, for g, being a non-Morse flow, there are three thin black edges in Γ which form a T-graph. Its edge corresponding to the stable manifold of a saddle which includes γ is called a *lower* edge, whereas the other its edges are called *upper* edges. This notion has already been used in [16], provided with appropriate illustrations.

An example of a distinguishing graph is depicted in Fig. 4. In this and other figures, the orientation of the outer boundary cycle is defined by counterclockwise motion, while the orientations of the inner boundary cycles are defined by clockwise motion.



(a) (b) (c) (d) (e)

Figure 4: Morse flow (a), its distinguishing graph (b), SC- (c), HSC_+- (d) and BSC- (e) flows

Theorem 3.1. *Two Morse flows* (SC-*flows,* BSC-*flows,* HSC_+-*flows,* HSC_--*flows*) *on compact connected surfaces are strongly topologically equivalent if and only if there exists an isomorphism between their distinguishing graphs that preserves the colors of vertices and edges, and either preserves or simultaneously reverses the orientations of all boundary cycles.*

The proof of this theorem is similar to the proof of [8, Classification theorem] and [16, Theorem 3.6].

If M is a closed surface and g is a Morse flow on it having only one source, then its distinguishing graph is nothing other than a chord diagram constructed in [6]. Moreover, if g is an SC-flow on a closed surface M, then its distinguishing graph is a T-diagram, studied in [16].

4 Topological classification of Morse flows through flow codes

Since now throughout this article, for any finite set L, an ordered multisubset $\{q_1, q_2, \ldots, q_m\}$ of L, $m \in \mathbb{N}$, written in a line provided that there may be a comma between any of its elements, will be called a *sequence of symbols $q_1 q_2 \cdots q_m$ from L.*

Suppose that we have an arbitrary fixed natural number N and a set of strings Λ_N being a union of the set { " {", "}" , " [", "] ", " 0 " } and a set of strings depicting the first N natural numbers. Hereafter in the paper, this notation will be fixed, as well.

Definition 4.1. Let us call a sequence of symbols $q_1 q_2 \cdots q_m$ from Λ_N a *Morse-like code list*, where $m \in \mathbb{N}$, if for the case where $N \geq 10$, and q_i and q_{i+1} depict numbers for $i = \overline{1, m-1}$, there is a comma between them; furthermore, one of next conditions holds true:

(1) $q_1 =$ " [" and $q_m =$ "] ", but $q_i \notin$ { "{", "}", " [", "] " } for $i = \overline{2, m-1}$, then we call the code list an *internal list*;

(2) q_1 = "{" and q_m = "}", but $q_i \notin$ {"{", "}"} for $i = \overline{2, m-1}$, then we call the code list a *boundary list*; moreover, next statements are true:

(a) the number of opening square brackets in the code list is equal to the number of closing ones in it;

(b) for any pair of square brackets in the code list that are both either opening or closing, there is always another square bracket between them.

If a boundary list contains square brackets, its cyclic permutation, where an opening square bracket is a second symbol, is called a *modified boundary list*. Each its subsequence of consecutive symbols starting from an opening square bracket and ending in the firstly met closing one is called its *sublist*. Internal lists and sublists of modified boundary lists will be called *source lists*.

Definition 4.2. We will call a sequence $L_1L_2\cdots L_m$, $m \in \mathbb{N}$, a *Morse-like code* if each L_i, $i = \overline{1,m}$, is a Morse-like code list and the first inclusions of natural numbers in the code form an increasing sequence. Such a code will be denoted as $L_1L_2\cdots L_m$.

Definition 4.3. A *cycle code* in a Morse-like code $L_1L_2\cdots L_m$ is a sequence $a_1a_2\cdots a_{n-1}$ of symbols from Λ_N, where a_1 and L_{q_1}, $a_1 \in L_{q_1}$, $q_1 = \overline{1,m}$, are fixed, a_1 is non-zero and:

(1) each a_i, $i = \overline{1,n}$, is a number considered with memorizing which code list it is in and where exactly in this code list it is placed;

(2) if $a_i \in L_{q_i}$, $i = \overline{1, n-1}$, then we move to the next number x in the list, where we have two options:

(a) if x = " 0 ", then $L_{q_{i+1}} = L_{q_i}$ and $a_{i+1} = x$;

(b) otherwise, we find its second location in the code, the obtained code list is $L_{q_{i+1}}$ and $a_{i+1} = x$;

(3) $a_1 = a_n$, but for any $i = \overline{2, n-1}$ $a_1 \neq a_i$.

A set of cycle codes, being a set of all cyclic permutations of the same cycle code, is called a *cycle*.

Suppose that we have a Morse flow g on a compact connected surface M with a fixed orientation and its distinguishing graph Γ with $N \in \mathbb{N}$ thin black edges.

We additionally fix a permutation $\xi : T \to T$ that maps one endpoint of a thin black edge in Γ to the other its endpoint.

Let $M_1, \ldots, M_n$ be boundary cycles of Γ. Having fixed for each $i = \overline{1,n}$ some point $v_i^1 \in M_i \cap V(\Gamma)$, we have $\{v_i^1, v_i^2, \ldots, v_i^{n_i}\}$, an ordered set of all vertices of Γ belonging to M_i obtained after bypassing M_i completely along the fixed orientation starting from the point v_i^1.

Thus we have a certain motion along each M_i, $i = \overline{1,n}$. We start bypassing each boundary cycle M_i according to the defined motion, gradually increasing i by 1. As we move, we meet vertices of Γ. Once we meet a point $v \in V(\Gamma) \cap M_i$, $i = \overline{1,n}$, we define the values of two functions in it. First, we define a mapping $h : V(\Gamma) \to \mathbb{N} \cup \{0\}$, which tells us the endpoints of how many thin black edges we have met since the beginning of bypassing M_1. Then we define a value of a function $f_{i,v_i^1} : \{v_i^1, v_i^2, \ldots, v_i^{n_i}\} \to \Lambda_N$ at the same point. A function f_{i,v_i^1}, $i = \overline{1,n}$, is defined according to the next rules:

(1) $f_{i,v_i^1}(v_i^n)$ = " 0 ", $n = \overline{1, n_i}$, if v_i^n has a white colour;

(2) $f_{i,v_i^1}(v_i^n)$ = " [", $n = \overline{1, n_i}$, if, while going through v_i^n, the color on M_i changes from the third colour to the second one;

(3) $f_{i,v_i^1}(v_i^n)$ = "] ", $n = \overline{1, n_i}$, if, while going through v_i^n, the color on M_i changes from the second colour to the third one;

(4) $f_{i,v_i^1}(v_i^n)$ corresponds to m, $m = \overline{1,N}$, $n = \overline{1, n_i}$, if $v_i^n \in T$ and one of the next conditions is true:

(a) $\xi(v_i^n) \notin M_1 \cup M_2 \cup \cdots \cup M_{i-1} \cup (M_i \cap \{v_i^1, v_i^2, \ldots, v_i^{n-1}\})$ and $h(v_i^n) = m$;

(b) $\xi(v_i^n) = v_j^l$, $1 \le j < i$, $1 \le l \le n_j$, where $f_{i,v_j^1}(v_j^l)$ corresponds to m;

(c) $\xi(v_i^n) = v_i^q$, $1 \le q < n$, where $f_{i,v_i^1}(v_i^q)$ corresponds to m.

Definition 4.4. A sequence of symbols

$$q_1 f_{i,v_i^1}(v_i^1) f_{i,v_i^1}(v_i^2) \cdots f_{i,v_i^1}(v_i^{n_i}) q_2$$

from Λ_N is called a *code list for M_i with a distinguished vertex v_i^1 for a flow g* if:

(1) $q_1 =$ "{" and $q_2 =$ "}" if $M_i \cap \partial M \neq \varnothing$, but $q_1 =$ "[" and $q_2 =$ "]", otherwise;

(2) if $N \ge 10$ and for some $j = \overline{1, n_i - 1}$ the symbols v_i^j and v_i^{j+1} depict the numbers, there is a comma put between them.

Definition 4.5. Let $x_1 \in M_1$ be an arbitrary fixed vertex of Γ. A sequence $\lambda_1 \lambda_2 \cdots \lambda_n$ is called a *Morse flow code with a distinguished vertex x_1 for a flow g* if each λ_i, $i = \overline{1, n}$, is a code list for a boundary cycle M_i' with a distinguished vertex $v_i' \in M_i'$ for the flow g so that:

(1) $M_1' = M_1$, $v_1' = x_1$;

(2) if we take the least non-zero number in $\lambda_1 \cdots \lambda_i$ for each $i = \overline{1, n-1}$ included in it only once, it will represent some $v \in T$, then $v_{i+1}' = \xi(v)$ and M_{i+1}' is a boundary cycle containing $\xi(v)$.

Such a Morse flow code will be denoted as $\lambda_1 \lambda_2 \cdots \lambda_n$.

Example 4.1. Let, on the distinguishing graph in Fig. 4, a vertex corresponding to the lower separatrix of the lower saddle be highlighted, with the outer boundary cycle oriented counterclockwise. Then the code looks like {12]0[}[123]{3}.

It is obvious that each code list for a boundary cycle in the distinguishing graph Γ is a Morse-like code list, and each Morse flow code is a Morse-like code. Each code list describes a situation on each boundary cycle in Γ, when each sublist tells us about what happens in a regular neighbourhood of each source. Besides, each cycle of the Morse flow code corresponds to the boundary of a 2-cell in the cellular decomposition generated by the distinguished graph of the Morse flow g.

Theorem 4.1. *There exists an isomorphism between the distinguishing graphs of two Morse flows that preserves the distinguished vertex, the colors of the vertices, the colors of the edges, and the orientations of boundary cycles if and only if they have the same codes.*

This theorem can be easily proved by the methods used in [9, Theorem 4.19] and [16, Theorems 3.9, 3.21].

Theorem 4.2. *Let g be a Morse flow on a compact surface M with its distinguishing graph Γ. Its Morse flow code $L_1 L_2 \cdots L_m$, $m \in \mathbb{N}$, with integers all depicted by symbols in Λ_N, has the following properties:*

1. *Each natural number, present in the code, appears there twice and at least once in some source list.*

2. *Each cycle code contains only one* 0.

3. *For any two code lists L_i and L_j in the code, $i < j$, $i, j = \overline{1, m}$, there exists a sequence of code lists starting from L_i and ending in L_j such that each pair of adjacent code lists in this sequence contains a couple of the same numbers placed in different code lists.*

4. *If V is the number of code lists and F is the number of cycles, then $V - N + F = 2$.*

Proof.

1. It is evident that each natural number appears in the code twice. If it is not in any source list, then there is a thin black edge in Γ whose endpoints, according to the construction, are saddles, but then there is an internal saddle connection in g, which is not possible for the Morse flows.

2. For a cycle code corresponding to a 2-cell of Γ with only one sink, it is impossible to have more than one 0 in the cycle code, or not have it at all.

3. This condition is equivalent to the connectivity of the distinguishing graph, which follows from the connectivity of the surface.

4. Collapsing each boundary cycle to a point, we obtain a sphere with its cellular decomposition. The 0-cells correspond bijectively to code lists, the 1-cells correspond bijectively to a set of natural numbers included in the code, the 2-cells correspond bijectively to cycles. Then for the Euler characteristic of the sphere, we have $\chi(S^2) = V - N + F = 2$. □

Theorem 4.3. *Let $L_1 L_2 \cdots L_m$, $m \in \mathbb{N}$, be a Morse-like code with integers all depicted by the symbols in Λ_N, and conditions $1-4$ from the previous theorem be true for $L_1 L_2 \cdots L_m$. Then there exists a Morse flow on a compact surface of genus χ for which $L_1 L_2 \cdots L_m$ serves as the Morse flow code so that χ is a number of code lists in the code that start and end in curly brackets.*

Proof. We form a distinguishing graph Γ for the Morse-like code $L_1 L_2 \cdots L_m$ and then construct a surface and a flow on it according to Γ, using the methods from [8] and [16]. □

5 Topological classification of codimension one gradient flows through flow codes

Suppose that M is a compact connected surface.

SN_+-flow (SN_--flow) code

Let g be an SN_+-flow on M (with a fixed orientation) with a saddle-node point sn. The flow g can be replaced by a Morse flow g' through a bifurcation which changes its behaviour in a regular neighbourhood U of sn from the one in Fig. 1 (b) to the one in Fig. 1 (a). Thus we have $p_0 \in U \cap \Omega_0(g')$ and $p_1 \in U \cap \Omega_1(g')$.

Let us fix $U_0 \subset U$, a regular neighbourhood for p_0.

Definition 5.1. A certain ordered list of symbols is called an SN_+-*flow code* for g if it has a form $\mathrm{SN}_+\mathrm{XXX}\cdots\mathrm{XX}$, where the part $\mathrm{XXX}\cdots\mathrm{XX}$ is a Morse flow code with a distinguished vertex $v \in \partial U_0 \cap W^s(p_1)$ for g'.

Now, let g be an SN_--flow with a saddle-node point sn defined on the same surface M with the same orientation. Reversing orientations of trajectories in g gives us an SN_+-flow g'.

Definition 5.2. A certain ordered list of symbols is called an SN_--*flow code* for g if it has a form $\mathrm{SN}_-\mathrm{XXX}\cdots\mathrm{XX}$ provided that $\mathrm{SN}_+\mathrm{XXX}\cdots\mathrm{XX}$ is an SN_+-flow code for g_1.

Example 5.1. Suppose that we have an SN_+-flow g defined on the sphere with 2 holes, so that it generates in the same way a Morse flow g_1 depicted in Fig. 4 (a). We assume that g is created through collapsing the trajectory α in g_1. It has the code $\mathrm{SN}_+[123]\{12]0[\}\{3\}$.

BSN_+-flow (BSN_--flow) code

Let g be a BSN_+-flow on M with a boundary saddle-node point sn. The flow g can be replaced by a Morse flow g' through a bifurcation which changes its behaviour in a regular neighbourhood U of bsn so that it changes from the one in Fig. 1 (e) to the one in Fig. 1 (d). Thus we have $p_0 \in U \cap \Omega_0(g')$ and $p_1 \in U \cap \Omega_1(g')$. The orientation of M is defined by the direction of the separatrix γ for which $\alpha(\gamma) = p_0$ and $\omega(\gamma) = p_1$.

Let us fix $U_0 \subset U$, a regular neighbourhood for p_0.

Definition 5.3. A certain ordered list of symbols is called a BSN_+-*flow code* for g if it has a form $\mathrm{BSN}_+\mathrm{XXX}\cdots\mathrm{XX}$, where the part $\mathrm{XXX}\cdots\mathrm{XX}$ is a Morse flow code with a distinguished vertex $v \in \partial U_0 \cap W^s(p_1)$ for g'.

Now let g be a BSN_--flow on M with a saddle-node point sn. Reversing orientations of trajectories in g gives us a BSN_+-flow g_1.

Definition 5.4. A certain ordered list of symbols is called a BSN_--*flow code* for g if it has a form $\mathrm{BSN}_-\mathrm{XXX}\cdots\mathrm{XX}$ provided that $\mathrm{BSN}_+\mathrm{XXX}\cdots\mathrm{XX}$ is a BSN_+-flow code for g_1.

Example 5.2. Suppose that we have a BSN_+-flow g defined on the sphere with 2 holes, so that it generates in the same way a Morse flow g_1 depicted in Fig. 4 (a). We assume that g is created through collapsing the trajectory β in g_1. It has the code BSN_+\{]10[2\}[12].

HS_+-flow (HS_--flow) code

Let g be an HS_+-flow on M (with a fixed orientation) with a degenerated saddle hs on ∂M. The flow g can be replaced by a Morse flow g' through a bifurcation which changes its behaviour in a regular neighbourhood U of hs so that it changes from the one in Fig. 2 (b) to the one in Fig. 2 (a). Thus we have $p_0 \in U \cap \Omega_0(g')$ and $p_1 \in U \cap \Omega_1(g')$.

Let us fix $U_0 \subset U$, a regular neighbourhood for p_0.

Definition 5.5. A certain ordered list of symbols is called an HS_+-*flow code* for g if it has a form $\mathrm{HS}_+\mathrm{XXX}\cdots\mathrm{XX}$, where the part $\mathrm{XXX}\cdots\mathrm{XX}$ is a Morse flow code with a distinguished vertex $v \in \partial U_0 \cap W^s(p_1)$ for g'.

Now, let g be an HS_--flow on M with a degenerated saddle hs on ∂M. Reversing orientations of trajectories in g gives us an HS_+-flow g_1.

Definition 5.6. A certain ordered list of symbols is called an HS_--*flow code* for g if it has a form $\mathrm{HS}_-\mathrm{XXX}\cdots\mathrm{XX}$ provided that $\mathrm{HS}_+\mathrm{XXX}\cdots\mathrm{XX}$ is an HS_+-flow code for g_1.

Example 5.3. Suppose that we have an HS_+-flow g defined on the sphere with 2 holes, so that it generates in the same way a Morse flow g_1 depicted in Fig. 4 (a). We assume that g is created through collapsing the trajectory φ in g_1. It has the code HS_+\{]10[2\}[12].

HN_+-flow (HN_--flow) code

Let g be an HN_+-flow on M (with a fixed orientation) with a degenerated node hn on ∂M. The flow g can be replaced by a Morse flow g' through a bifurcation which changes its behaviour in a regular neighbourhood U of hn so that it changes from the one in Fig. 2 (e) to the one in Fig. 2 (d). Thus we have $p_0 \in U \cap \Omega_0(g')$ and $p_1 \in U \cap \Omega_1(g')$.

Let us fix $U_1 \subset U$, a regular neighbourhood for p_1.

Definition 5.7. A certain ordered list of symbols is called an HN_+-*flow code* for g if it has a form $\mathrm{HN}_+\mathrm{XXX}\cdots\mathrm{XX}$, where the part $\mathrm{XXX}\cdots\mathrm{XX}$ is a Morse flow code with a distinguished vertex $v \in \partial U_1 \cap W^s(p_1)$ for g'.

Now, let g be an HN_--flow on M with a degenerated node hn on ∂M. Reversing orientations of trajectories in g gives us an HN_+-flow g_1.

Definition 5.8. A certain ordered list of symbols is called an HN_--*flow code* for g if it has a form $\mathrm{HN}_-\mathrm{XXX}\cdots\mathrm{XX}$ provided that $\mathrm{HN}_+\mathrm{XXX}\cdots\mathrm{XX}$ is an HN_+-flow code for g_1.

Example 5.4. Suppose that we have an HN_+-flow g defined on the sphere with 2 holes, so that it generates in the same way a Morse flow g_1 depicted in Fig. 4 (a). We assume that g is created through collapsing the trajectory ω in g_1. It has the code HN_+\{10[2]\}[12].

BDS-flow code

Let g be a BDS-flow on M with a double saddle bds on ∂M. The flow g can be replaced by a Morse flow g' through a bifurcation which changes its behaviour in a regular neighbourhood U of bsn so that it changes from the one in Fig. 3 (b) to the one in Fig. 3 (a). Thus we have $p_0 \in U \cap \Omega_1(g')$ and $p_1 \in U \cap \Omega_1(g')$. The orientation of M is defined by the direction of the separatrix γ for which $\alpha(\gamma) = p_0$ and $\omega(\gamma) = p_1$.

Let us fix $U_0 \subset U$, a regular neighbourhood for p_0.

Definition 5.9. A certain ordered list of symbols is called a BDS-*flow code* for g if it has a form BDSXXX$\cdots$XX, where the part XXX$\cdots$XX is a Morse flow code with a distinguished vertex $v \in \partial U_0 \cap W^s(p_1)$ for g'.

Example 5.5. Suppose that we have a BDS-flow g defined on the sphere with 2 holes, so that it generates in the same way a Morse flow g_1 depicted in Fig. 4. We assume that g is created through collapsing the trajectory η in g_1. It has the code BDS{1[2]0}[12].

SC-flow code and HSC$_+$-flow (HSC$_-$-flow) code

Suppose that we have an arbitrary fixed natural number N, an SC-flow (an HSC$_+$-flow) g on M with a fixed orientation and its distinguishing graph Γ with one T-graph formed by thin black edges and $N \in \mathbb{N}$ other thin black edges.

We additionally fix a permutation $\xi : T \to T$ which maps each vertex in the T-graph into another its vertex of degree 1 obtained through bypassing the T-graph along the orientation fixed on M_1, but maps any other endpoint of a thin black edge in Γ to the other its endpoint.

Let $M_1, \dots, M_n$ be boundary cycles of Γ. As we have done it in the case with Morse flows, we define a motion on each boundary cycle, thus having for each $i = \overline{1,n}$ an ordered set $\{v_i^1, v_i^2, \dots, v_i^{n_i}\} \subset M_i$.

We start bypassing each boundary cycle M_i according to the defined motion, gradually increasing i by 1. As we move, we meet vertices of Γ. Once we meet a point $v \in V(\Gamma) \cap M_i$, $i = \overline{1,n}$, we will define the value of two functions in it. First, we define a mapping $h : V(\Gamma) \to \mathbb{N} \cup \{0\}$ which tells us endpoints of how many thin black edges beyond the T-graph we have met since the beginning of bypassing M_1. Then we define a value of a function $f_{i,v_i^1} : \{v_i^1, v_i^2, \dots, v_i^{n_i}\} \to \Lambda_{N+2}$ at the same point. A function f_{i,v_i^1}, $i = \overline{1,n}$, is defined according to the next rules:

(1) $f_{i,v_i^1}(v_i^n) =$ " 0 ", $n = \overline{1,n_i}$, if v_i^n has a white colour;

(2) $f_{i,v_i^1}(v_i^n) =$ " [", $n = \overline{1,n_i}$, if while going through v_i^n, the color on M_i changes from the third colour to the second one;

(3) $f_{i,v_i^1}(v_i^n) =$ "] ", $n = \overline{1,n_i}$, if while going through v_i^n, the color on M_i changes from the second colour to the third one;

(4) $f_{i,v_i^1}(v_i^n) =$ " 1 ", $n = \overline{1,n_i}$, if v_i^n is an endpoint of a lower edge of the T-graph;

(5) $f_{i,v_i^1}(v_i^n) =$ " 2 ", $n = \overline{1,n_i}$, if v_i^n is an endpoint of an upper edge of the T-graph;

(6) $f_{i,v_i^1}(v_i^n)$ corresponds to $m+2$, $m = \overline{1,N}$, $n = \overline{1,n_i}$, if $v_i^n \in T$, v_i^n is not a vertex of the T-graph in Γ, and one of the next conditions is true:

   (a) $\xi(v_i^n) \notin M_1 \cup M_2 \cup \cdots \cup M_{i-1} \cup (M_i \cap \{v_i^1, v_i^2, \dots, v_i^{n-1}\})$ and $h(v_i^n) = m$;

   (b) $\xi(v_i^n) = v_j^l$, $1 \le j < i$, $1 \le l \le n_j$, where $f_{i,v_j^1}(v_j^l)$ corresponds to $m+2$;

   (c) $\xi(v_i^n) = v_i^q$, $1 \le q < n$, where $f_{i,v_i^1}(v_i^q)$ corresponds to $m+2$.

The definition for a code list of a boundary cycle in Γ with a distinguished vertex on it is similar to the one for a Morse flow.

Definition 5.10. Let $x_1 \in M_1$ be an endpoint of a lower edge of the T-graph in Γ. A sequence $\mathrm{SC}\lambda_1\lambda_2\cdots\lambda_n$ ($\mathrm{HSC}_+\lambda_1\lambda_2\cdots\lambda_n$) is called an SC-*flow* (HSC$_+$-*flow*) *code* for g if each λ_i, $i=\overline{1,n}$, is a code list for a boundary cycle M_i' with a distinguished vertex $v_i' \in M_i'$ for the flow g so that:

(1) $M_1' = M_1$, $v_1' = x_1$;

(2) distinguished vertices on the other boundary cycles are defined according to the next rules:

  (a) if we take the least number greater than 1 in $\lambda_1\cdots\lambda_i$ for each $i=\overline{1,n-1}$ included in it only once, it will represent some $v \in T$, and then $v_{i+1}' = \xi(v)$ and M_{i+1}' is a boundary cycle containing $\xi(v)$;

  (b) if there is no number from the previous item, then $v_{i+1}' = \xi(v_1')$ and M_{i+1}' is a boundary cycle containing $\xi(v_1')$.

Now, let g be an HSC$_-$-flow on M with a saddle connection belonging to $\mathrm{Int}(M)$. Reversing orientations of trajectories in g gives us an HSC$_+$-flow g_1.

Definition 5.11. A certain ordered list of symbols is called an HSC$_-$-*flow code* for g if it has a form $\mathrm{HSC}_-\mathrm{XXX}\cdots\mathrm{XX}$ provided that $\mathrm{HSC}_+\mathrm{XXX}\cdots\mathrm{XX}$ is an HSC$_+$-flow code for g_1.

Example 5.6. Suppose that we have an SC-flow g defined on the sphere with 2 holes. Its separatrix skeleton can be seen in Fig. 4 (c). The code is SC{1]0[2}{2]0[}.

Example 5.7. Suppose that we have an HSC$_+$-flow g defined on the sphere with 2 holes. Its separatrix skeleton can be seen in Fig. 4 (d). The code is HSC$_+${1}{2]0[2}.

BSC-**flow code**

Let g be a BSC-flow on M with a saddle connection $\gamma \subset \mathrm{Int}(M)$ on M, provided M has a fixed orientation.

Definition 5.12. A certain ordered list of symbols is called a BSC-*flow code* for g if it has a form BSCXXX$\cdots$XX, where the part XXX$\cdots$XX is a sequence of symbols obtained from g according to the same algorithm used for obtaining a Morse flow code for a Morse flow on M, except for the next facts:

(1) $\alpha(\gamma)$ is considered to be a distinguished vertex and $\beta(\gamma)$ is marked in code by number 1;

(2) after bypassing the first boundary cycle, one must move to the boundary cycle which contains $\beta(\gamma)$;

(3) if a vertex belongs to T, but it is not an endpoint of γ, then its number defined by the algorithm should increase by 1.

Example 5.8. Suppose that we have a BSC-flow g defined on the sphere with 2 holes. Its separatrix skeleton can be seen in Fig. 4 (e). The code is BSC{[]}{10[]}.

The structure of codimension one gradient flows

For BSN$_-$-, BSN$_+$ and BDS-flows the appropriate flow codes are defined unambiguously through giving a fixed orientation and a distinguished vertex.

Nevertheless, in the case of SN$_-$-, SN$_+$-, SC-, HN$_-$-, HN$_+$-, HS$_-$-, HS$_+$-, HSC$_-$-, HSC$_+$ and BSC-flows, we fix just the distinguished vertex, having two ways to define the orientation on the surface. That's why each flow of one of these types has actually a pair of flow codes which we call *symmetric*.

The theorem below follows immediately from the construction of flow codes for all these types of flows.

Theorem 5.1. BSN$_-$-, BSN$_+$ *and BDS-flows are strongly topologically equivalent if and only if their codes are equal.* SN$_-$-, SN$_+$-, SC-, HN$_-$-, HN$_+$-, HS$_-$-, HS$_+$-, HSC$_-$-, HSC$_+$ *and* BSC-*flows are strongly topologically equivalent if and only if they have the same (unordered) pairs of their symmetric codes.*

Fix the following sets of strings:

$O_1 := \{$“SN_-”, “SN_+”, “BSN_-”, “BSN_+”, “HS_-”, “HS_+”, “HN_-”, “HN_+”, “BDS”, “BSC”$\}$,

$O_2 := \{$“SC”, “HSC_-”, “HSC_+”$\}$.

Definition 5.13. Let N be an arbitrary fixed natural number. We call a concatenation M of one arbitrary string s from $O_1 \cup O_2$ and a sequence of symbols L from Λ_N a 1-*code* if the following statements are true:

(1) if $s \in O_1$, then L is a Morse-like code;

(2) if $s \in O_2$, then L is a concatenation of Morse-like code lists, where first inclusions of natural number greater than 2 form an increasing sequence.

A sequence L is called a *kernel* of a 1-code M.

For a kernel of any 1-code notions of internal lists, boundary lists, modified boundary lists, sublists and source lists, given in the previous section, are still relevant. If condition (1) is true, the notions of cycle code and cycle, given in the previous section, are still relevant, as well.

Definition 5.14. A *cycle code* in a kernel $L_1L_2\cdots L_m$ of a 1-code satisfying property 2) from the previous definition is a sequence $a_1a_2\cdots a_{n-1}$ of symbols from Λ_N, where a_1 and L_{q_1}, $a_1 \in L_{q_1}$, $q_1 = \overline{1,m}$, are fixed, a_1 is non-zero and:

(1) each a_i, $i = \overline{1,n}$, is a number, considered with memorizing in which code list it is placed and where exactly in this code list it is;

(2) if $a_i \in L_{q_i}$, $i = \overline{1,n-1}$, then we move to the next number x in the list, where we have two options:

  (a) if $x =$ “ 0 ”, then $L_{q_{i+1}} = L_{q_i}$ and $a_{i+1} = x$;

  (b) if $x =$ “ 1 ”, then a_{i+1} is “ 2 ” emerging first in the kernel, and the obtained code list is $L_{q_{i+1}}$; if x is “ 2 ” emerging first in the kernel, then a_{i+1} is “ 2 ” emerging second in the kernel, and the obtained code list is $L_{q_{i+1}}$; if x is “ 2 ” emerging second in the kernel, then $a_{i+1} =$ “ 1 ”, and the obtained code list is $L_{q_{i+1}}$;

  (c) otherwise, we find its second location in the code, that is, the obtained code list is $L_{q_{i+1}}$ and $a_{i+1} = x$;

  (3) $a_1 = a_n$, but for any $i = \overline{2,n-1}$, $a_1 \neq a_i$.

A set of cycle codes, being a set of all cyclic permutations of the same cycle code, is called a *cycle*.

From the above-listed properties of 1-codes and flow codes for codimension one gradient flows on compact connected surfaces, we obtain the following corollaries.

Corollary 5.1. *Each flow code Λ for a codimension one gradient flow g on M has the following properties:*

1. *Λ starts with one of the following fragments:*

$$SN_+[1,\ SN_-[1,\ BSN_+\{],\ BSN_-\{],\ HS_+\{1,\ HS_-\{1,\ HN_+\{1,\ HN_-\{1,\ BDS\{1,$$
$$SC\{1,\ SC[1,\ HSC_+\{1,\ HSC_-\{1,\ BSC\{1$$

2. *Properties 2–4 from Theorem 4.2 hold true for the kernel of Λ; property 1 from Theorem 4.2 holds true for the kernel of Λ if Λ starts neither with* SC, *nor with* HSC, *nor with* BSC.

3. *If Λ starts with* SC, HSC *or* BSC, *then* 1 *appears in it once, whereas any other natural number, present in Λ, appears there twice.*

4. *If Λ starts with* SC, *then each natural number, present in it, appears in some its source list at least once. If Λ starts with* HSC *or* BSC, *then each natural number, present in it and greater than* 1, *appears in some its source list at least once.*

Corollary 5.2. *If a* 1*-code satisfies the properties 1–4 from the previous theorem, then there exists a codimension one gradient flow on some compact connected surface having this* 1*-code as its flow code.*

Conclusion

All possible structures of Morse flows and codimension one gradient flows on spheres with holes containing no more than 6 singular points have been found (Table 1). We hope that the research conducted in this work can be extended to other surfaces and to non-gradient flows.

Table 1: Numbers of bifurcations of flows with n singular points

surface, n	SN_+	BSN_+	HS_+	HN_+	BDS	SC	HSC_+	BSC	$\sum$
D^2, 3	1	1	1	1	0	0	0	0	8
D^2, 4	2	1	2	4	0	0	0	0	18
D^2, 5	13	10	4	6	3	0	1	0	71
D^2, 6	31	11	13	24	9	5	4	1	181
$S^1 \times I$, 4	0	0	2	2	0	0	0	1	9
$S^1 \times I$, 5	4	7	6	4	1	0	1	1	46
$S^1 \times I$, 6	12	2	20	24	7	2	5	5	140
$F_{0,3}$, 6	0	0	5	2	0	0	0	2	16

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