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O. Pravdyvyi, A. Stanzhytskyi, Y. Perestyuk

APPROXIMATION OF INFINITE DIMENSION NEUTRAL-TYPE STOCHASTIC DIFFERENTIAL EQUATION BY SYSTEMS WITHOUT DELAY

Abstract. In this article we consider the approximation of an infinite-dimensional neutral-type stochastic differential equation by systems without delay. Continuity modulo Lemma is proved for neutral type stochastic delay evolution equation in a Hilbert space. An approximation system of stochastic differential equations without delay is proposed for neutral type stochastic delay evolution equation.

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რეზიუმე. სტატიაში განხილულია ნეიტრალური ტიპის უსასრულოგანზომილებიანი სტოქასტური დიფერენციალური განტოლების აპროქსიმაცია დაგვიანების გარეშე სისტემებით. დამტკიცებულია მოდულით უწყვეტობის ლემა ჰილბერტის სივრცეში ნეიტრალური ტიპის სტოქასტური ევოლუციური დაგვიანებული განტოლებისთვის. ნეიტრალური ტიპის სტოქასტური ევოლუციური დაგვიანებული განტოლებისთვის შემოთავაზებულია დაგვიანების არმქონე სტოქასტურ დიფერენციალურ განტოლებებათა აპროქსიმაციული სისტემა.

1 Introduction

The main goal of this work is to study an approximation systems for neutral type stochastic delay equations in Hilbert space of the form

$$d\big(u(t) - g(u(t), u(t-h))\big) = (Au(t) + f(u(t-h), u(t)))\,dt + \sigma(u(t-h), u(t))\,dW(t), \quad t \in [0, T], \tag{1.1}$$

$$u(t) = \phi(t), \quad t \in [-h, 0]. \tag{1.2}$$

Here, A is an infinitesimal generator of a strong continuous semigroup $\{S(t), t \geq 0\}$ of bounded linear operators on separable Hilbert space H. The random noise $W(t)$ is a Q-Wiener process on a separable Hilbert space K. For some $h > 0$, we denote by $C_h := C([-h, 0], H)$ the space of continuous H-valued functions $\phi : [-h, 0] \to H$ with a norm

$$\|\phi\|_{C_h} := \sup_{\theta \in [-h,0]} \|\phi(\theta)\|,$$

where $\|\cdot\|$ stands for the norm in H. The solution $u(t)$ of (1.1) we consider in a sense of a mild solution. The solution of (1.1) is sometimes referred as a *state process*. We also denote $u_t := u(t+\theta)$, where $\theta \in [-h, 0)$. The functions f, g map $H \times H$ to H, and $\sigma : H \to L_2^0$, where $L_2^0 = L(Q^{\frac{1}{2}}K, H)$ is the space of Hilbert–Shmidt operators from $Q^{\frac{1}{2}}K$ to H. Finally, $\phi : [-h, 0] \times \Omega \to H$ is the initial function, where $(\Omega, \mathcal{F}, P)$ is a complete probability space.

In this article, the proposed approach consists of splitting the delay interval and constructing the corresponding approximation system. It is important to note that the number of equations in this system grows depending on the number of subintervals. The main result of this article shows that if the number of subintervals tends to infinity, then the mean square distance between the solution of equation (1.1)–(1.2) and the solutions of the approximation system tends to zero.

This work is a continuation of [7, 8]. In both articles, the authors considered a similar problem in the non-neutral case ($g = 0$): in the first article, for the finite-dimensional case, and in the second article, for the infinite-dimensional case. This article, like [7, 8], is a generalization of the works of I. M. Cherevko and his students [2–4]. In these works, the initial problem for a system of delay equations is replaced by a set of Cauchy problems, constructed in a specific way from the original system. This approach is based on the M. M. Krasovsky's ideas on expanding the solution of a delay system into a Taylor series with respect to the length of the delay interval $h > 0$.

It should be noted that the presence of a delay under the derivative (neutral type) complicates the study significantly. Indeed, in the definition of a mild solution (see Definition 2.1 below) there appear two additional terms: $g(u^h(t-h))$ and $\int_0^t AS(t-s)g(u^h(t-s))\,ds$. The compact semigroup $S(t)$ does not act on the first term, so establishing of compactness of the family $g(u^h(t-h,\omega))$ in space $C([-h, 0], H)$ is not obvious. For the integral term it should be noted that, generally speaking, if $g(\phi) \in H$, then this term has a non-integrable singularity at $s = t$. We overcome this difficulty by introducing fractional powers of the operator $(-A)$. In particular, we show that if g is regular enough to $g(u^h(s-h)) \in D((-A)^\alpha)$ for $\alpha \in (0, 1)$, then this singularity becomes integrable.

The paper is organized as follows. In Section 2, we introduce the necessary notations and preliminary results. Section 3 is devoted to the main results of our work: continuity modulo Lemma and an approximation system for equation (1.1)–(1.2) with equations without delay.

2 Preliminaries

Throughout this paper, H and K are separable Hilbert spaces with the norms $\|\cdot\|$ and $\|\cdot\|_K$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and Q be a linear bounded covariance operator such that $tr(Q) < \infty$. Introduce

$$W(t) := \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\beta_k(t) e_k, \quad t \geq 0,$$

which is a Q-Wiener process on $t \geq 0$. Here, $\beta_k(t)$ are standard, one-dimensional, independent Wiener processes, $\{e_k, k \geq 1\}$ is an orthonormal system in K, and a sequence of real nonnegative numbers λ_k satisfies

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, \dots,$$

and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

Also, let $\{\mathcal{F}_t, t \geq 0\}$ be a normal filtration satisfying:

1. $W(t)$ is $\mathcal{F}_t$-measurable;
2. $W(t+h) - W(t)$ is independent of $\mathcal{F}_t$ for all $h \geq 0$ and $t \geq 0$.

Let $U_0 = Q^{\frac{1}{2}}(K)$ and let $L_2^0 = L_2(U_0, H)$ be the space of all Hilbert–Schmidt operators from U_0 to H with the inner product $(\Phi, \Psi)_{L_2^0} = tr[\Phi Q \Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively.

We assume that A is the infinitesimal generator of an analytic semigroup $S(t) = e^{At}$ of bounded in H operators. From [1] it is equivalent to the fact that $(-A)$ is a sectorial operator.

We assume that $S(t)$ is semigroup of compact operators, hence from Theorem 3.2 [6] it is continuous in the uniform operator topology.

From [6] we can deduce that for all $\alpha \in (0, 1)$ the fractional power of operator $(-A)$ is a closed linear operator with domain $D(-A^\alpha)$.

Denote by H^α the Hilbert space $D(-A)^\alpha$ endowed with the norm

$$\|u\|_\alpha := \|(-A)^\alpha u\|.$$

To ensure the existence and uniqueness of solution, we have to impose additional conditions on the operator A and mappings f, σ, g:

(1) If $\sigma(-A)$ is a spectrum of $(-A)$, we have

$$\operatorname{Re} \sigma(-A) > \delta > 0,$$

and A generates a semigroup of compact operators $S(t)$ in H.

(2) For all $u, v, u_1, v_1 \in H$, we have

$$\|f(u, v) - f(u_1, v_1)\| \leq L\big(\|u - u_1\| + \|v - v_1\|\big)$$

and

$$\|\sigma(u, v) - \sigma(u_1, v_1)\|_{L_2^0} \leq L\big(\|u - u_1\| + \|v - v_1\|\big)$$

for some $L > 0$.

(3) There exists $\alpha \in (\frac{1}{2}, 1)$ such that for all $u, v, u_1, v_1 \in H$, the function g satisfies

$$\|g(u, v) - g(u_1, v_1)\|_\alpha \leq M_g\big(\|u - u_1\| + \|v - v_1\|\big)$$

for some $M_g \in (0, \frac{1}{2\sqrt{2}})$.

It is easy to see that condition (2) implies linear growth of f, σ in H and condition (3) implies linear growth of g in H_α.

In the sequel, we will use the following theorem from [1].

Theorem 2.1 ([1, Th 1.4.3])**.** *There exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha S(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t}, \quad t > 0.$$

In particular,

$$\|S(t)\| \leq C_0 e^{-\delta t}, \quad t > 0.$$

We introduce a mild solution of equation (1.1)–(1.2) as follows.

Definition 2.1. A continuous $\mathcal{F}_t$ adapted stochastic process $u : [-h,T]\times\Omega\to H$ is called a mild solution to (1.1)–(1.2) for $t\in[0,T]$ if it satisfies the integral equation

$$u(t)=S(t)(\phi(0)-g(\phi(-h)))+g(u(t),u(t-h))-\int_0^t AS(t-s)g(u(s-h))\,ds$$
$$+\int_0^t S(t-s)f(u(s-h),u(s))\,ds+\int_0^t S(t-s)\sigma(u(s-h),u(s))\,dW(s),$$

and $u(t)=\phi(t)$ a.s. for $t\in[-h,0]$.

It follows from [5] and [9] that, if conditions (1)–(3) hold, the initial equation (1.1)–(1.2) has a unique on $[0,T]$ mild solution in the sense of Definition 2.1, with a bounded p-th moment ($p\geq 1$).

In the next section, we present the main results of this article.

3 Approximation of delay stochastic equation by an equation without delay

The following lemma will play an important role in the subsequent proof.

Lemma 3.1 (Continuity modulo Lemma)**.** *Under conditions* (1)–(3)*, for the solution* $u(t)$ *of equation* (1.1)–(1.2) *the following is true:*

$$\sup_{t_1\in[-h,T]}\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(t_1)\|^2\leq C(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0, \tag{3.1}$$

Proof. From Definition 2.1 we have

$$\|u(t)\|^2\leq 8\,\|S(t)(\phi(0)-g(\phi(-h)))\|^2$$
$$+2\,\|g(u(t),u(t-h))\|^2+8\left\|\int_0^t AS(t-s)g(u(s),u(s-h))\,ds\right\|^2$$
$$+8\left\|\int_0^t S(t-s)f(u(s),u(s-h))\,ds\right\|^2+8\left\|\int_0^t S(t-s)\sigma(u(s),u(s-h))\,dW(s)\right\|^2,$$

which can be rewritten as

$$\|u(t)\|^2\leq 8\,\|S(t)(\phi(0)-g(\phi(-h)))\|^2+8\,\|g(u(t),u(t-h))\|^2+I_1+I_2+I_3.$$

Now, we can estimate the terms, starting from $8\,\|S(t)(\phi(0)-g(\phi(-h)))\|^2$:

$$8\,\|S(t)(\phi(0)-g(\phi(-h)))\|^2\leq 8\,C_0\|\phi(0)-g(\phi(-h))\|^2.$$

Next, for $8\,\|g(u(t),u(t-h))\|^2$, we have

$$2\,\|g(u(t),u(t-h))\|^2\leq 4\,M_g^2(\|u(t)\|^2+\|u(t-h)\|^2),$$

hence

$$\mathbf{E}\sup_{s\in[0,t]}4\,M_g^2\big(\|u(t)\|^2+\|u(t-h)\|^2\big)$$
$$\leq 4\,M_g^2\,\mathbf{E}\Big(\sup_{s\in[0,t]}(\|u(t)\|^2)+\sup_{s\in[0,t]}(\|u(t-h)\|^2)\Big)\leq 8\,M_g^2\,\mathbf{E}\sup_{s\in[0,t]}(\|u(t)\|^2)+4\,M_g^2\|\phi\|_{C_h}^2$$

Next, for the term I_1,

$$8\left\|\int_0^t AS(t-s)\,g(u(s),u(s-h))\,ds\right\|^2$$

$$\le 16\,M_g^2\int_0^t C_{1-\alpha}^2(t-s)^{2\alpha-2}e^{-2\delta(t-s)}\,ds\cdot\int_0^t\left(\|u(s)\|^2+\|u(s-h)\|^2\right)ds$$

$$\le 16\,M_g^2 C(T)\int_0^t \sup_{s_1\in[0,s]}\left(\|u(s_1)\|^2+\sup_{s_1\in[0,s]}\|u(s_1-h)\|^2\right)ds$$

$$\le 32\,M_g^2 C(T)\int_0^t \sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,M_g^2 TC(T)\|\phi\|_{C_h}^2,$$

which means that

$$\mathbf{E}\,32\,M_g^2C(T)\int_0^t\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,M_g^2TC(T)\|\phi\|_{C_h}^2$$
$$\le 32\,M_g^2C(T)\int_0^t\mathbf{E}\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,M_g^2TC(T)\|\phi\|_{C_h}^2.$$

Next, for I_2,

$$8\left\|\int_0^t S(t-s)f(u(s),u(s-h))\,ds\right\|^2\le 32\,C_0T\int_0^t\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,C_0T^2\|\phi\|_{C_h}^2,$$

hence

$$\mathbf{E}\,32\,C_0T\int_0^t\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,C_0T^2\|\phi\|_{C_h}^2\le 32\,C_0T\int_0^t\mathbf{E}\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,C_0T^2\|\phi\|_{C_h}^2.$$

Next, for I_3, we have

$$8\left\|\int_0^t S(t-s)\sigma(u(s),u(s-h))\,dW(s)\right\|^2$$
$$\le 8\int_0^t C_0^2\|\sigma(u(s),u(s-h))\|_{L_2^0}^2\,ds\le 32\,C_0T\int_0^t\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,C_0T^2\|\phi\|_{C_h}^2,$$

so, we can write

$$\mathbf{E}\,32\,C_0T\int_0^t\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,C_0T^2\|\phi\|_{C_h}^2\le 32\,C_0T\int_0^t\mathbf{E}\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds+16\,C_0T^2\|\phi\|_{C_h}^2.$$

Collecting all estimations, we get

$$\mathbf{E}\sup_{s\in[0,t]}\|u(s)\|^2 \le 8\,C_0\|\phi(0)-g(\phi(-h))\|^2 + 8\,M_g^2\,\mathbf{E}\sup_{s\in[0,t]}(\|u(t)\|^2)$$
$$+\,4\,M_g^2\|\phi\|_{C_h}^2 + 32\,M_g^2 C(T)\int_0^t \mathbf{E}\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds + 16\,M_g^2 TC(T)\|\phi\|_{C_h}^2$$
$$+\,64\,C_0 T\int_0^t \mathbf{E}\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds + 32\,C_0T^2\|\phi\|_{C_h}^2.$$

From condition (3) it follows that $M_g < \frac{1}{2\sqrt{2}}$, therefore, we can write

$$\mathbf{E}\sup_{s\in[0,t]}\|u(s)\|^2 \le 8\,C_0C_{M_g}\|\phi(0)-g(\phi(-h))\|^2 + 4\,M_g^2C_{M_g}\|\phi\|_{C_h}^2$$
$$+\,32\,(M_g^2+2\,C_0)C(T)C_{M_g}\int_0^t \mathbf{E}\sup_{s_1\in[0,s]}\|u(s_1)\|^2\,ds + 16\,(M_g^2+2\,C_0)TC(T)C_{M_g}\|\phi\|_{C_h}^2,$$

where $C_{M_g} = \frac{1}{1-8\,M_g^2}$.

So, from the Grownwall lemma, we get the estimate

$$\mathbf{E}\sup_{s\in[0,t]}\|u(s)\|^2 \le K(T,\|\phi\|_{C_h}^2).$$

Consequently,

$$\mathbf{E}\sup_{s\in[-h,T]}\|u(s)\|^2 \le K(T,\|\phi\|_{C_h}^2).$$

If $t_1+l<0$, we have

$$\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(t_1)\|^2 = \mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|\phi(t_2)-\phi(t_1)\|^2 = C_1(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0, \tag{3.2}$$

which follows from the fact, that ϕ is uniformly continuous.

Next, if $-l<t_1<0$, then

$$\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(t_1)\|^2 = \mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-\phi(t_1)\|^2$$
$$\le 2\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(0)\|^2 + 2\,\|\phi(t_1)-\phi(0)\|^2$$
$$\le 2\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(0)\|^2 + 2\,C_1(T,\|\phi\|_{C_h},l).$$

Considering the fact that $l\to 0$, it follows that for every $C>0$, $l<C$, we can, for example, set $C:=h$. Therefore, we get

$$\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(0)\|^2$$
$$= 8\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\left(\left\|S(t_2)(\phi(0)-g(\phi(-h)))-S(0)(\phi(0)-g(\phi(-h)))\right\|^2\right)$$
$$+\,2\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\left\|g(u(t_2),\phi(t_2-h))-g(\phi(0),\phi(-h))\right\|^2$$
$$+\,16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\left\|\int_0^{t_2} AS(t_2-s)g(u(s),u(s-h))\,ds\right\|^2$$

$$+\,16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big\|\int_0^{t_2} S(t_2-s)f(u(s-h),u(s))\,ds\Big\|^2$$
$$+\,16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big\|\int_0^{t_2} S(t_2-s)\sigma(u(s-h),u(s))\,dW(s)\Big\|^2,$$

which can be rewritten as

$$\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(0)\|^2$$
$$=8\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big(\big\|S(t_2)(\phi(0)-g(\phi(-h)))-S(0)(\phi(0)-g(\phi(-h)))\big\|^2\Big)$$
$$+\,2\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\big\|g(u(t_2),\phi(t_2-h))-g(\phi(0),\phi(-h))\big\|^2+I_4+I_5+I_6.$$

The first term $8\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}(\|S(t_2)(\phi(0)-g(\phi(-h)))-S(0)(\phi(0)-g(\phi(-h)))\|^2)$ converges to zero from the uniform continuity of the operator $S(t)u$.

For the second term, we can do the following:

$$2\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\big\|g(u(t_2),\phi(t_2-h))-g(\phi(0),\phi(-h))\big\|^2$$
$$\le 4\,M_g^2\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big(\|u(t_2)-\phi(0)\|^2+\|\phi(t_2-h)-\phi(-h)\|^2\Big).$$

We can estimate I_4 as follows:

$$16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big\|\int_0^{t_2} AS(t_2-s)g(u(s),u(s-h))\,ds\Big\|^2$$
$$\le 16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big(\int_0^{t_2}\|A^{1-\alpha}S(t_2-s)\|\cdot\|g(u(s),u(s-h))\|_\alpha\,ds\Big)^2$$
$$\le 16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big(\int_0^{t_2} C_{\alpha-1}(t_2-s)^{1-\alpha}e^{-\delta(t_2-s)}\cdot M_g\big(1+\|u(s-h)\|+\|u(s)\|\big)\,ds\Big)^2$$
$$\le 64\,M_g^2\sup_{r\in[0,l]}\int_0^{t_1+r} C_{\alpha-1}^2(t_1+r-s)^{2-2\alpha}e^{-2\delta(t_1+r-s)}\,ds\cdot\int_0^{t_1+l}\mathbf{E}\sup_{s_1\in[0,s]}\|u(s)\|^2\,ds$$
$$\le 64\,M_g^2C_{1-\alpha}^2\sup_{r\in[0,l]}\int_0^{t_1+r}(t_1+r-s)^{2\alpha-2}\,ds\cdot\int_0^{t_1+l}\mathbf{E}\sup_{s_1\in[0,s]}\|u(s)\|^2\,ds$$
$$=64\,M_g^2C_{1-\alpha}^2\sup_{r\in[0,l]}\big((t_1+r)^{2\alpha-1}-(t_1+r-t_1-r)^{2\alpha-1}\big)\cdot\int_0^{t_1+l}\mathbf{E}\sup_{s_1\in[0,s]}\|u(s)\|^2\,ds$$
$$=64\,M_g^2C_{1-\alpha}^2(t_1+l)^{2\alpha-1}\int_0^{t_1+l}\mathbf{E}\sup_{s_1\in[0,s]}\|u(s)\|^2\,ds$$
$$\le 64\,M_g^2C(l)K(T,\|\phi\|_{C_h})=C_1(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0,$$

if $\alpha>\frac{1}{2}$.

Next, for I_5,

$$16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big\|\int_0^{t_2} S(t_2-s)f(u(s-h),u(s))\,ds\Big\|^2$$
$$\le 32\,L^2\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\int_0^{t_2}\|S(t_2-s)\|^2\,ds\cdot\int_0^{t_2}\big(1+\|u(s)\|^2+\|u(s-h)\|^2\big)\,ds$$
$$\le 64\,L^2\mathbf{E}\sup_{r\in[0,l]}\int_0^{t_1+r}\|S(t_1+r-s)\|^2\,ds\cdot\int_0^{t_1+l}\mathbf{E}\sup_{s_1\in[0,s]}\|u(s)\|^2\,ds = C_3(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0.$$

For I_6, we have

$$16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big\|\int_0^{t_2} S(t_2-s)\sigma(u(s-h),u(s))\,dW(s)\Big\|^2$$
$$\le 16\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\int_0^{t_2}\|S(t_2-s)\|^2\cdot\|\sigma(u(s-h),u(s))\|^2_{L_2^0}\,ds$$
$$\le 32\,L^2\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\int_0^{t_2}\|S(t_2-s)\|^2\,ds\cdot\int_0^{t_2}\big(1+\|u(s)\|^2+\|u(s-h)\|^2\big)\,ds$$
$$\le 64\,L^2\mathbf{E}\sup_{r\in[0,l]}\int_0^{t_1+r}\|S(t_1+r-s)\|^2\,ds\cdot\int_0^{t_1+l}\mathbf{E}\sup_{s_1\in[0,s]}\|u(s)\|^2\,ds = C_4(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0.$$

Thus, there exists $C_5(T,\|\phi\|_{C_h},l)\to 0$, as $l\to 0$, such that

$$\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_2)-u(t_1)\|^2\le C_5(T,\|\phi\|_{C_h},l) \tag{3.3}$$

for $-l<t_1<0$.

Let us split the interval $[0,T]$ by h. Thus there exists some $N=[\frac{T}{h}]$ such that the sequence $h_n=nh$ is the following: $0\le h_1<h_2<\dots<h_{N-1}<h_N=T$.

Now, we assume that $t_1\in[0,h]$, therefore, we have the following.

Let $t_2=t_1+r$, then, from Definition 2.1, for all $t_1\ge 0$, we have

$$\begin{aligned}u(t_1+r)-u(t_1) &= S(t_1+r)(\phi(0)-g(\phi(-h)))\\ &\quad - S(t_1)(\phi(0)-g(\phi(-h)))+g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\\ &\quad -\int_0^{t_1+r}AS(t_1+r-s)g(u(s),u(s-h))\,ds+\int_0^{t_1}AS(t_1-h)g(u(s),u(s-h))\,ds\\ &\quad +\int_0^{t_1+r}S(t_1+r-s)f(u(s-h),u(s))\,ds-\int_0^{t_1}S(t_1-s)f(u(s-h),u(s))\,ds\\ &\quad +\int_0^{t_1+r}S(t_1+r-s)\sigma(u(s-h),u(s))\,dW(s)-\int_0^{t_1}S(t_1-s)\sigma(u(s-h),u(s))\,dW(s).\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\|u(t_1+r)-u(t_1)\|^2\\
&\quad\le 8\left\|S(t_1+r)(\phi(0)-g(\phi(-h)))-S(t_1)(\phi(0)-g(\phi(-h)))\right\|^2\\
&\qquad+2\,\mathbf{E}\sup_{r\in[0,l]}\left\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\right\|^2\\
&\qquad+8\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1+r}AS(t_1+r-s)g(u(s),u(s-h))\,ds-\int_0^{t_1}AS(t_1-h)g(u(s),u(s-h))\,ds\right\|^2\\
&\qquad+8\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1+r}S(t_1+r-s)f(u(s-h),u(s))\,ds-\int_0^{t_1}S(t_1-s)f(u(s-h),u(s))\,ds\right\|^2\\
&\qquad+8\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1+r}S(t_1+r-s)\sigma(u(s-h),u(s))\,dW(s)-\int_0^{t_1}S(t_1-s)\sigma(u(s-h),u(s))\,dW(s)\right\|^2,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\mathbf{E}\sup_{r\in[0,l]}\|u(t_1+r)-u(t_1)\|^2\\
&\quad\le 8\left\|S(t_1+r)(\phi(0)-g(\phi(-h)))-S(t_1)(\phi(0)-g(\phi(-h)))\right\|^2\\
&\qquad+2\,\mathbf{E}\sup_{r\in[0,l]}\left\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\right\|^2\\
&\qquad+16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1}\big(AS(t_1+r-s)-AS(t_1-s)\big)g(u(s),u(s-h))\,ds\right\|^2\\
&\qquad+16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_{t_1}^{t_1+r}AS(t_1+r-s)g(u(s),u(s-h))\,ds\right\|^2\\
&\qquad+16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1}(S(t_1+r-s)-S(t_1-s))f(u(s-h),u(s))\,ds\right\|^2\\
&\qquad+16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_{t_1}^{t_1+r}S(t_1-s+r)f(u(s-h),u(s))\,ds\right\|^2\\
&\qquad+16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1}(S(t_1+r-s)-S(t_1-s))\sigma(u(s-h),u(s))\,dW(s)\right\|^2\\
&\qquad+16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_{t_1}^{t_1+r}S(t_1-s+r)\sigma(u(s-h),u(s))\,dW(s)\right\|^2.
\end{aligned}$$

We denote

$$\begin{aligned}
&\mathbf{E}\sup_{r\in[0,l]}\|u(t_1+r)-u(t_1)\|^2=8\left\|S(t_1+r)(\phi(0)-g(\phi(-h)))-S(t_1)(\phi(0)-g(\phi(-h)))\right\|^2\\
&\quad+2\,\mathbf{E}\sup_{r\in[0,l]}\left\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\right\|^2+J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}$$

First of all, we can estimate $2\,\mathbf{E}\sup\limits_{r\in[0,l]}\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\|^2$ as follows:

$$
\begin{aligned}
&2\,\mathbf{E}\sup_{r\in[0,l]}\big\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\big\|^2\\
&\qquad\le 4\,M_g\,\mathbf{E}\sup_{r\in[0,l]}\Big(\|u(t_1+r-h)-u(t_1-h)\|^2+\|u(t_1+r)-u(t_1)\|^2\Big)\\
&\qquad\qquad= 4\,M_g\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\Big(\|u(t_2-h)-u(t_1-h)\|^2+\|u(t_2)-u(t_1)\|^2\Big).
\end{aligned}
$$

It is easy to see that $-l<t_1-h<0$, hence t_2-h can be either lesser than 0 or in $[0,l]$. If $t_2-h<0$, than from (3.2),

$$
4\,M_g\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\big(\|u(t_2-h)-u(t_1-h)\|^2\big)\longrightarrow 0\ \text{ as }\ l\to 0.
$$

If $t_2-h\in[0,l]$ then from (3.3) it follows that

$$
4\,M_g\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\big(\|u(t_2-h)-u(t_1-h)\|^2\big)\longrightarrow 0\ \text{ as }\ l\to 0.
$$

Thus, there exists $K_1(T,\|\phi\|_{C_h})\to 0$ as $l\to 0$ such that

$$
4\,M_g\,\mathbf{E}\sup_{t_2\in[t_1,t_1+l]}\big(\|u(t_2-h)-u(t_1-h)\|^2\big)\le K_1(T,\|\phi\|_{C_h}). \tag{3.4}
$$

Next, for J_1,

$$
\begin{aligned}
&16\,\mathbf{E}\sup_{r\in[0,l]}\Big\|\int_0^{t_1}\big(AS(t_1+r-s)-AS(t_1-s)\big)g(u(s),u(s-h))\,ds\Big\|^2\\
&\le 16\,\mathbf{E}\sup_{r\in[0,l]}\Big(\int_0^{t_1}\Big\|S\Big(\frac{t_1-s}{2}+r\Big)-S\Big(\frac{t_1-s}{2}\Big)\Big\|\cdot\Big\|A^{1-\alpha}S\Big(\frac{t_1-s}{2}\Big)\Big\|\cdot\|g(u(s),u(s-h))\|_\alpha\,ds\Big)^2\\
&\qquad\le 16\,TM_gK(T,\|\phi\|_{C_h})\sup_{r\in[0,l]}\int_0^{t_1}\Big\|S\Big(\frac{t_1-s}{2}+r\Big)-S\Big(\frac{t_1-s}{2}\Big)\Big\|^2\cdot\Big\|A^{1-\alpha}S\Big(\frac{t_1-s}{2}\Big)\Big\|^2\,ds.
\end{aligned}
$$

We denote

$$
\int_0^{t_1}\Big\|S\Big(\frac{t_1-s}{2}+r\Big)-S\Big(\frac{t_1-s}{2}\Big)\Big\|^2\cdot\Big\|A^{1-\alpha}S\Big(\frac{t_1-s}{2}\Big)\Big\|^2\,ds=\psi(r).
$$

It is easy to see that the function $\psi(r)$ exists, is continuous and as $r\to 0$, $\psi(r)\to 0$ since the semigroup $S(t)$ is compact. Assume $\lim\limits_{l\to 0}\sup\limits_{r\in[0,l]}\psi(r)>0$. This means that there exist the sequence $\{r_n|n\ge 1\}$ and $\{C_n>0|n\ge 1\}$ such that $\sup\limits_{r\in[0,r_n]}\psi(r)=C_n$ and $C_n\to C>0$ as $n\to\infty$. From the continuity, we can deduce that there exists a point $r_n^*\in[0,r_n]$ such that $\sup\limits_{r\in[0,r_n]}\psi(r)=\psi(r_n^*)$, therefore, there exists the sequence $\{r_n^*|n\ge 1\}$ on which $\psi(r)$ converges to C as $r_n^*\to 0$, which contradicts convergence to zero. Hence there exist $C_7(T,\|\phi\|^2_{C_h},l)\to 0$ as $l\to 0$ such that

$$
16\,\mathbf{E}\sup_{r\in[0,l]}\Big\|\int_0^{t_1}\big(AS(t_1+r-s)-AS(t_1-s)\big)g(u(s),u(s-h))\,ds\Big\|^2\le C_7(T,\|\phi\|^2_{C_h},l).
$$

The term J_2 can be estimated analogously to I_4.

Next, for J_3,

$$16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1}\big(S(t_1+r-s)-S(t_1-s)\big)f(u(s-h),u(s))\,ds\right\|^2$$

$$16\,\mathbf{E}\sup_{r\in[0,l]}\left(\int_0^{t_1}\|S(t_1+r-s)-S(t_1-s)\|\cdot\|f(u(s-h),u(s))\|\,ds\right)^2$$

$$\le 16\,TLK(T,\|\phi\|_{C_h})\sup_{r\in[0,l]}\int_0^{t_1}\|S(t_1+r-s)-S(t_1-s)\|^2\,ds\le C_8(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0.$$

The term J_4 can be estimated analogously to I_5.

Next, for J_5,

$$16\,\mathbf{E}\sup_{r\in[0,l]}\left\|\int_0^{t_1}(S(t_1+r-s)-S(t_1-s))\sigma(u(s-h),u(s))\,dW(s)\right\|^2$$

$$16\,\mathbf{E}\sup_{r\in[0,l]}\int_0^{t_1}\big\|S(t_1+r-s)-S(t_1-s)\big\|^2\cdot\|\sigma(u(s-h),u(s))\|^2_{L_2^0}\,ds$$

$$\le 16\,TLK(T,\|\phi\|_{C_h})\sup_{r\in[0,l]}\int_0^{t_1}\|S(t_1+r-s)-S(t_1-s)\|^2\,ds\le C_9(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0.$$

The term J_6 can be estimated analogously to I_6.

Thus for $t_1\in[0,h_1]$, we have

$$\mathbf{E}\sup_{r\in[0,l]}\|u(t_1+r)-u(t_1)\|^2\le C_{10}(T,\|\phi\|_{C_h},l)\longrightarrow 0,\quad l\to 0.$$

Next, for $t_1\in[h_1,h_2]$ we can do the following:

$$\begin{aligned}\|u(t_1+r)-u(t_1)\|^2 &= 8\,\big\|(S(t_1+r)-S(t_1))\cdot(\phi(0)-g(\phi(-h)))\big\|^2\\
&+2\,\big\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\big\|^2\\
&+16\left\|\int_0^{t_1}\big(AS(t_1+r-s)-AS(t_1-s)\big)g(u(s),u(s-h))\,ds\right\|^2\\
&+16\left\|\int_{t_1}^{t_1+r}AS(t_1+r-s)g(u(s),u(s-h))\,ds\right\|^2\\
&+16\left\|\int_0^{t_1}(S(t_1+r-s)-S(t_1-s))f(u(s-h),u(s))\,ds\right\|^2\\
&+16\left\|\int_{t_1}^{t_1+r}S(t_1-s+r)f(u(s-h),u(s))\,ds\right\|^2\\
&+16\left\|\int_0^{t_1}(S(t_1+r-s)-S(t_1-s))\sigma(u(s-h),u(s))\,dW(s)\right\|^2\\
&+16\left\|\int_{t_1}^{t_1+r}S(t_1-s+r)\sigma(u(s-h),u(s))\,dW(s)\right\|^2.\end{aligned}$$

The term $2\sup\limits_{r\in[0,l]}\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\|^2$ can be estimated as follows:

$$2\sup_{r\in[0,l]}\big\|g(u(t_1+r),u(t_1+r-h))-g(u(t_1),u(t_1-h))\big\|^2 \le 4\,M_g^2\sup_{r\in[0,l]}\Big(\|u(t_1+r)-u(t_1)\|^2+\|u(t_1+r-h)-u(t_1-h)\|^2\Big).$$

Using analogous reasoning to (3.4), we can deduce that there exists $K_2(T,\|\phi\|_{C_h})\to 0$ as $l\to 0$ such that

$$4\,M_g^2\sup_{r\in[0,l]}\big(\|u(t_1+r-h)-u(t_1-h)\|^2\big)\le K_2(T,\|\phi\|_{C_h}). \tag{3.5}$$

Estimations of all other terms are analogous to the previous step. For every step, we introduce $K_n(T,\|\phi\|_{C_h})$ in the form (3.5) dependant only on the estimation of previous step. Therefore, from the fact that $N<\infty$, we can deduce that there exists $C(T,\|\phi\|_{C_h},l)\to 0$ as $l\to 0$ such that (3.1) holds and Lemma is proven. □

Introduce the following system:

$$\begin{cases} d\big(z_0(t)-g(z_0(t),z_m(t))\big)=\big(Az_0+f(z_0(t),z_m(t))\big)\,dt+\sigma(z_0(t),z_m(t))\,dW(t),\\ dz_j(t)=\dfrac{m}{h}\,(z_{j-1}(t)-z_j(t)),\quad t\in[0,T],\\ z_j(0)=\phi\Big(-\frac{hj}{m}\Big),\quad j=0,\dots,m. \end{cases} \tag{3.6}$$

Definition 3.1. System (3.6) is called an approximating system for (1.1)–(1.2) in the mean square if

$$\sup_{t\in[0,T]}\mathbf{E}\Big\|u\Big(t-\frac{h}{m}j\Big)-z_j(t)\Big\|^2\longrightarrow 0,\quad m\to\infty,\quad j=0,\dots,m. \tag{3.7}$$

Theorem 3.1 (Approximation system)**.** *Under the conditions* (1)–(3)*, system* (3.6) *is an approximating system for* (1.1)–(1.2) *in the mean square in the sense of Definition 3.1.*

Proof. Denote

$$N_j(t)=\mathbf{E}\Big\|u\Big(t-\frac{hj}{m}\Big)-z_j^{(1)}(t)\Big\|^2,\quad j=0,\dots,m. \tag{3.8}$$

Similar to [8, Sec. 3], using Lemma 3.1, we get for all $j=1,\dots,m$ the following inequality:

$$\sup_{t\in[0,T]}\sqrt{N_j(t)}\le\alpha\Big(T,\|\phi\|_{C_h},\frac{h}{m}\Big)+\sqrt{\sup_{t\in[0,T]}N_0(t)} \tag{3.9}$$

where $\alpha(T,\|\phi\|_{C_h},\frac{h}{m})$ is some constant dependant on T, $\|\phi\|_{C_h}$ and m such that $\alpha(T,\|\phi\|_{C_h},\frac{h}{m})\to 0$ as $m\to\infty$.

Consequently, for $j=m$,

$$\sup_{t\in[0,T]}\sqrt{\mathbf{E}\,\|u(t-h)-z_m(t)\|^2}\le\alpha\Big(T,\|\phi\|_{C_h},\frac{h}{m}\Big)+\sqrt{\sup_{t\in[0,T]}\mathbf{E}\,\|u(t)-z_0(t)\|^2}\,.$$

Now, we can estimate N_0 as follows:

$$\begin{aligned} z_0(t)-u(t)&=\big(g(z_0(t),z_m(t))-g(u(t),u(t-h))\big)\\ &\quad+\int_0^t AS(t-s)\big(g(u(s),u(s-h))-g(z_0(s),z_m(s))\big)\,ds\\ &\quad+\int_0^t S(t-s)\big(f(z_0(s),z_m(s))-f(u(s),u(s-h))\big)\,ds\\ &\quad+\int_0^t S(t-s)\big(\sigma(z_0(s),z_m(s))-\sigma(u(s),u(s-h))\big)\,dW(s). \end{aligned}$$

Therefore, we get

$$\sup_{t\in[0,T]} \mathbf{E}\,\|z_0(t)-u(t)\|^2 \le 2\sup_{t\in[0,T]} \mathbf{E}\,\big\|\big(g(z_0(t),z_m(t))-g(u(t),u(t-h))\big)\big\|^2$$
$$+6\sup_{t\in[0,T]} \mathbf{E}\left\|\int_0^t AS(t-s)\big(g(u(s),u(s-h))-g(z_0(s),z_m(s))\big)\,ds\right\|^2$$
$$+6\sup_{t\in[0,T]} \mathbf{E}\left\|\int_0^t S(t-s)\big(f(z_0(s),z_m(s))-f(u(s),u(s-h))\big)\,ds\right\|^2$$
$$+6\sup_{t\in[0,T]} \mathbf{E}\left\|\int_0^t S(t-s)\big(\sigma(z_0(s),z_m(s))-\sigma(u(s),u(s-h))\big)\,dW(s)\right\|^2.$$

Next, we can estimate the terms on the right-hand side in following way.

For the term $2\sup_{t\in[0,T]} \mathbf{E}\,\|(g(z_0(t),z_m(t))-g(u(t),u(t-h)))\|^2$, we have

$$2\sup_{t\in[0,T]} \mathbf{E}\,\big\|\big(g(z_0(t),z_m(t))-g(u(t),u(t-h))\big)\big\|^2$$
$$\le 2\,M_g^2 \sup_{t\in[0,T]} \mathbf{E}\left(\|z_0(t)-u(t)\|^2+\|z_m(t-h)-u(t-h)\|^2\right)$$
$$\le 6M_g^2 \sup_{t\in[0,T]} \mathbf{E}\,(\|z_0(t)-u(t)\|^2)+4\,M_g^2\alpha^2\Big(T,\|\phi\|_{C_h},\frac{h}{m}\Big).$$

Next, for the term $6\sup_{t\in[0,T]} \mathbf{E}\,\|\int_0^t AS(t-s)\big(g(u(s),u(s-h))-g(z_0(s),z_m(s))\big)\,ds\|^2$, we can do the following:

$$6\sup_{t\in[0,T]} \mathbf{E}\left\|\int_0^t AS(t-s)\big(g(u(s),u(s-h))-g(z_0(s),z_m(s))\big)\,ds\right\|^2$$
$$\le 6\int_0^T C_{2-2\alpha}(t-s)^{2\alpha-2}e^{-\delta(t-s)}\,ds\cdot\int_0^T \mathbf{E}\,\big\|\big(g(u(s),u(s-h))-g(z_0(s),z_m(s))\big)\big\|_\alpha^2\,ds$$
$$\le 6C(T,\alpha)\cdot\int_0^T 3M_g^2\sup_{t\in[0,T]} \mathbf{E}\,(\|z_0(t)-u(t)\|^2)+2\,M_g^2\alpha^2\Big(T,\|\phi\|_{C_h},\frac{h}{m}\Big)\,ds$$
$$=18\,C(T,\alpha)M_g^2\int_0^T \sup_{t\in[0,T]} \mathbf{E}\,(\|z_0(t)-u(t)\|^2)\,ds+12\,M_g^2TC(T,\alpha)\alpha^2\Big(T,\|\phi\|_{C_h},\frac{h}{m}\Big).$$

Here, $C(T,\alpha)<\infty$ if $\alpha>\frac{1}{2}$.

Next, to estimate

$$6\sup_{t\in[0,T]} \mathbf{E}\left\|\int_0^t S(t-s)\big(f(z_0(s),z_m(s))-f(u(s),u(s-h))\big)\,ds\right\|^2,$$

we can do the following:

$$6\sup_{t\in[0,T]} \mathbf{E}\left\|\int_0^t S(t-s)\big(f(z_0(s),z_m(s))-f(u(s),u(s-h))\big)\,ds\right\|^2$$

$$\le 6T \int_0^T \mathbf{E} \big\| \big(f(u(s), u(s-h)) - f(z_0(s), z_m(s)) \big) \big\|^2 \, ds$$

$$\le 18\, T L^2 \int_0^T \sup_{t\in[0,T]} \mathbf{E}\big(\|z_0(t) - u(t)\|^2\big) \, ds + 12\, L^2 T^2 \alpha^2 \Big(T, \|\phi\|_{C_h}, \frac{h}{m} \Big).$$

From the property on Ito integral we can get similar estimate for $6 \sup\limits_{t\in[0,T]} \mathbf{E} \| \int\limits_0^t S(t-s)(\sigma(z_0(s), z_m(s)) - \sigma(u(s), u(s-h)))\, dW(s)\|^2$:

$$6 \sup_{t\in[0,T]} \mathbf{E} \Big\| \int_0^t S(t-s)\big(\sigma(z_0(s), z_m(s)) - \sigma(u(s), u(s-h))\big)\, dW(s) \Big\|^2$$

$$\le 18\, T L^2 \int_0^T \sup_{t\in[0,T]} \mathbf{E}\, (\|z_0(t) - u(t)\|^2) \, ds + 12\, L^2 T^2 \alpha^2 \Big(T, \|\phi\|_{C_h}, \frac{h}{m} \Big).$$

Hence, we get

$$\sup_{t\in[0,T]} \mathbf{E}\, \|z_0(t) - u(t)\|^2 \le 6 M_g^2 \sup_{t\in[0,T]} \mathbf{E}\, (\|z_0(t) - u(t)\|^2)$$

$$+ 18 (T L^2 + C(T,\alpha) M_g^2) \int_0^T \sup_{t\in[0,T]} \mathbf{E}\, (\|z_0(t) - u(t)\|^2)\, ds$$

$$+ 12\, T (L^2 T + M_g^2 C(T,\alpha)) \alpha^2 \Big(T, \|\phi\|_{C_h}, \frac{h}{m} \Big).$$

From condition (3), we can deduce that $M_g \le \sqrt{\frac{1}{6}}$ and write

$$\sup_{t\in[0,T]} \mathbf{E}\, \|z_0(t) - u(t)\|^2 \le 18\, C_M (T L^2 + C(T,\alpha) M_g^2) \int_0^T \sup_{t\in[0,T]} \mathbf{E}\, (\|z_0(t) - u(t)\|^2)\, ds$$

$$+ 12\, T C_M (L^2 T + M_g^2 C(T,\alpha)) \alpha^2 \Big(T, \|\phi\|_{C_h}, \frac{h}{m} \Big),$$

where $C_M = \frac{1}{1 - 6 M_g^2}$.

Using Gronwall lemma, we have

$$\sup_{t\in[0,T]} \mathbf{E}\, \|z_0(t) - u(t)\|^2$$

$$\le 12\, T C_M (L^2 T + M_g^2 C(T,\alpha)) \alpha^2 \Big(T, \|\phi\|_{C_h}, \frac{h}{m} \Big) \cdot e^{18\, C_M T (T L^2 + C(T,\alpha) M_g^2)} \longrightarrow 0, \quad m \to \infty.$$

Therefore, taking into consideration (3.8) and (3.9), we find that condition (3.7) holds, hence theorem is proves. □

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Authors' addresses:

O. Pravdyvyi
Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska Str., Kyiv 01601, Ukraine
E-mail: awxrvtb@gmail.com

A. Stanzhytskyi
Institute of Mathematics of National Academy of Sciences of Ukraine, 3 Tereschenkivska Str., Kyiv 01024, Ukraine
E-mail: a.stanzhytskyi@gmail.com

Y. Perestyuk
Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska Str., Kyiv 01601, Ukraine
E-mail: peretyuk@gmail.com