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Ihor Cherevko, Andrii Dorosh, Svitlana Ilika, Dmytro Shkilniuk

APPROXIMATION OF LINEAR DIFFERENTIAL-FUNCTIONAL EQUATIONS

Abstract. Schemes for approximating solutions of initial value problems for linear differential-functional equations by solutions of Cauchy problems for special systems of ordinary linear differential equations are proposed and substantiated.

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რეზიუმე. შემოთავაზებული და დასაბუთებულია წრფივი ფუნქციონალურ-დიფერენციალური განტოლებებისთვის საწყისი ამოცანების ამონახსნების აპროქსიმაციის სქემა ჩვეულებრივ წრფივ დიფერენციალურ განტოლებათა სპეციალური სისტემებისთვის კოშის ამოცანების ამონახსნების საშუალებით.

1 Introduction

Differential-difference and differential-functional equations arise when accounting for delays in the description of various technical, physical, and natural processes. Equations of this type occur in the modeling of ecological and biological systems, in studies of nuclear reactor behavior, in systems with semiconductor elements, in chemical-technological processes, and in other phenomena that depend on their history. This is why the systems of differential-functional equations have become a relevant subject of research [3, 14].

Currently, there are no universal methods for finding exact solutions to differential-functional equations, which makes the development and justification of algorithms for obtaining approximate solutions to initial and boundary value problems for such equations an important and relevant task. Of particular interest are studies that enable the application of methods from the theory of ordinary differential equations to analyse differential–functional equations.

The methods for approximating differential-functional equations by ordinary dynamical systems were proposed in [4, 9]. The accuracy of approximation of nonlinear differential-difference equations with delay was investigated in [13] in the case where the solution of the initial value problem is differentiable or satisfies the Lipschitz condition. Further studies on approximation schemes for differential-difference equations in spaces of continuous functions on a finite interval were carried out in the works of I. M. Cherevko and L. A. Piddubna [1, 2].

Analysis of the accuracy of approximating the vector delay element for various input functions, as well as the generalization of approximation schemes for systems of differential-difference equations of delay and neutral types, was considered in the works of I. M. Cherevko and O. V. Matvii [10, 11].

The application of approximation schemes for differential-difference equations to the approximate determination of non-asymptotic roots of quasipolynomials and to the stability analysis of solutions of systems of linear differential equations with delay was studied in [7, 8, 12].

In this work, we construct and justify approximation schemes for linear differential-functional equations with delay by sequences of special systems of ordinary differential equations.

2 Problem statement, approximation scheme

Let $C = C([\alpha,\beta], R^n)$ be the space of continuous functions which map $[\alpha,\beta]$ to R^n with the norm $\|\varphi\| = \sup\limits_{\alpha<\theta<\beta} |\varphi(\theta)|$.

For an arbitrary continuous function $x(t)$ defined on $[-\tau, T]$, $T, \tau > 0$, and an arbitrary fixed $t \in [0,T]$ we denote by x_t [5, 6] the following function: $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$.

Let us consider the initial problem for a differential-functional equation of the form

$$\begin{aligned} \frac{dx(t)}{dt} &= L(t, x_t), \quad t \in [0,T], \\ x_0 &= \varphi, \end{aligned} \tag{2.1}$$

where $L(t,\varphi)$ is a linear functional on φ that can be represented by a Stieltjes integral $L(t,\varphi) = \int\limits_{-\tau}^{0} [d\eta(t,\theta)]\varphi(\theta)$, where $\eta(t,\theta)$ is an $n\times n$ matrix, whose elements are the functions of bounded variation on θ for each t and continuous on t uniformly with respect to θ.

Definition ([5])**.** We call the function $x(t)$ a solution of the initial problem (2.1) if

(1) $x_t \in C$, $t \in [-\tau, T]$,

(2) $x_0 = \varphi(\theta)$, $\theta \in [-\tau, 0]$,

(3) x_t satisfies equation (2.1) for $t \in [0,T]$.

We consider equation (2.1), where the linear functional $L(t,\varphi)$ has the form

$$L(t,\varphi) = \sum_{i=0}^{p} A_i(t)\varphi(-\tau_i) + \int\limits_{-\tau}^{0} D(t,\theta)\varphi(\theta)\, d\theta, \tag{2.2}$$

often encountered in applied applications, $A_i(t)$, $i = \overline{0,p}$, are $n \times n$ continuous matrix functions, $D(t,\theta)$ is an $n \times n$ matrix function, whose components $d_{ij}(\tau,\theta)$ are continuous functions over the set of variables on $[0,T] \times [-\tau,\theta]$, $0 = \tau_0 < \tau_1 < \cdots < \tau_p = \tau$.

Let us consider the schemes for approximating the initial problem (2.1) by a sequence of Cauchy problems for systems of ordinary differential equations, which are similar to the approximation schemes for delay differential-difference equations.

In the case of a linear functional of the form (2.2), we associate the initial problem (2.1) with a system of ordinary differential equations

$$\begin{aligned} \frac{dz_0(t)}{dt} &= \sum_{i=0}^{p} A_i(t) z_{l_i}(t) + \frac{\tau}{m} \sum_{i=0}^{m-1} D\Big(t, -\frac{\tau(m-K)}{m}\Big) z_{m-i}(t), \\ \frac{dz_j(t)}{dt} &= \frac{m}{\tau}\,(z_{j-1}(t) - z_j(t)), \;\; j = \overline{1,m}, \;\; t \in [0,T], \end{aligned} \tag{2.3}$$

with the initial conditions

$$z_j(0) = \varphi\Big(-\frac{\tau j}{m}\Big), \;\; j = \overline{0,m}, \tag{2.4}$$

where the indices l_j are defined by the equalities $l_j = [\frac{m\tau_j}{\tau}]$, $j = \overline{0,p}$, and the second term in the first equation of (2.3) is obtained by replacing the integral with the formula of left rectangles with a step $h = \dfrac{\tau}{m}$.

If the integral is replaced by the trapezoidal quadrature formula with a step $h = \frac{\tau}{m}$, then we associate with the initial problem (2.1) a Cauchy problem for a system of ordinary differential equations of the form

$$\frac{dz_0(t)}{dt} = \sum_{i=0}^{p} A_i(t) z_{l_i}(t) + \frac{\tau}{2m}\big(D(t,-\tau) z_m + D(t,0) z_0(t)\big) + \frac{\tau}{m} \sum_{i=1}^{m-1} D\Big(t, -\frac{\tau(m-i)}{m}\Big) z_{m-i}(t), \tag{2.5}$$

$$\begin{aligned} \frac{dz_j(t)}{dt} &= \frac{m}{\tau}\,(z_{j-1}(t) - z_j(t)), \;\; j = \overline{1,m}, \;\; t \in [0,T], \\ z_j(0) &= \varphi\Big(-\frac{\tau j}{m}\Big), \;\; j = \overline{0,m}. \end{aligned} \tag{2.6}$$

We will say that the solutions of the Cauchy problems (2.3), (2.4) and (2.5), (2.6) approximate the solution of the initial problem (2.1) if the following relations are satisfied:

$$\Big\| x\Big(t - \frac{\tau j}{m}\Big) - z_j(t) \Big\| \to 0, \;\; j = \overline{0,m}, \;\; t \in [0,T].$$

3 Approximation of the delay element

A simpler and fairly common method for solving delay system synthesis problems is to replace the delay element with several aperiodic elements. This replacement allows us to obtain qualitatively similar results if the delay value of τ is small.

In cases where the delay τ is not insignificant, the sequence of delay elements that are sequentially connected to each other is considered in order to improve the approximation.

Let us consider $m \in \mathbb{N}$ delay elements generated by some input function $x(t)$

$$y_1(t) = x\Big(t - \frac{\tau}{m}\Big), \; y_2(t) = x\Big(t - \frac{2\tau}{m}\Big), \; \ldots, \; y_m(t) = x(t-\tau), \;\; x \in \mathbb{R}^n, \;\; t \in [0,T]. \tag{3.1}$$

Output functions $y_j(t)$ of delay elements will be determined if their initial state $x(t) = \varphi(t)$, $t \in [-\tau, 0]$, is specified.

Let us assume that the input function $x(t)$ is continuous on $[-\tau, T]$.

We associate the delay elements (3.1) with a sequence of aperiodic elements described by a system of ordinary differential equations of the form

$$\begin{aligned}&\frac{\tau}{m}\,\frac{dz_1(t)}{dt}+z_1(t)=x(t),\\&\frac{\tau}{m}\,\frac{dz_j(t)}{dt}+z_j(t)=z_{j-1}(t),\quad j=\overline{2,m},\quad t\in[0,T].\end{aligned}\tag{3.2}$$

Consider the initial values of aperiodic elements and the initial states of delay elements to be the same

$$z_j(0)=y_j(0)=x\Big(-\frac{j\tau}{m}\Big),\quad j=\overline{1,m}.\tag{3.3}$$

We will investigate the deviation between the functions $y_j(t)$ and $z_j(t)$, $t\in[0,T]$, $j=\overline{1,m}$, in the case of a continuous input function $x(t)$.

Note that system (3.2), (3.3) is studied in [13] in the case where the function $x(t)$ is scalar and satisfies the Lipschitz condition with a constant K_1 or has a bounded derivative on the section $[-\tau,T]$. In this case, the following inequalities are proven to be true:

$$|z_j(t)-y_j(t)|\le\frac{4K_1\tau}{\sqrt{m}}\,,\quad t\in[0,T],\quad j=\overline{1,m}.$$

The case where the scalar input function $x(t)\in C[-\tau,T]$ is considered in [13]. It has been established that in this case the following inequalities are true:

$$|z_j(t)-y_j(t)|\le 2\Big(\frac{K\tau}{\sqrt{m}}+\omega\Big(x,\frac{\tau}{\sqrt{m}}\Big)\Big),\quad t\in[0,T],\quad j=\overline{1,m},\tag{3.4}$$

where the constant $K>0$ does not depend on m, and $\omega(x,\frac{\tau}{m})=\max\limits_{|t'-t''|<\frac{\tau}{m},\,t',t''\in[-\tau,T]}|x(t')-x(t'')|$ is the continuity modulus of the function $x(t)$ on $[-\tau,T]$.

Let us examine the approximation accuracy of the vector delay element in the case where the input function $x:[-\tau,T]\to\mathbb{R}^n$ in the system (3.2), (3.3) is continuous.

Let $x(t)=(x_1(t),\dots,x_n(t))$, $z_j(t)=(z_{j1}(t),\dots,z_{jn}(t))$, $j=\overline{1,m}$. Then system (3.2), (3.3) in a coordinate form has the following representation:

$$\begin{aligned}&\frac{\tau}{m}\,\frac{dz_{1i}(t)}{dt}+z_{1i}(t)=x_i(t),\\&\frac{\tau}{m}\,\frac{dz_{ji}(t)}{dt}+z_{ji}(t)=z_{j-1,i}(t),\quad j=\overline{2,m},\quad i=\overline{1,n},\quad t\in[0,T],\end{aligned}\tag{3.5}$$

$$z_{ji}(0)=x_i\Big(-\frac{j\tau}{m}\Big),\quad j=\overline{1,m},\quad i=\overline{1,n}.\tag{3.6}$$

Let us consider the smoothed functions

$$x_i^{(1)}(t)=\frac{1}{h}\int\limits_t^{t+h}x_i(s)\,ds,\quad t\in[-\tau,T],\quad i=\overline{1,n}\tag{3.7}$$

(we continue the functions $x_i(t)$ on a segment $[T,T+h]$ continuously, as constant).

We evaluate the difference $x_i^{(2)}(t)=x_i(t)-x_i^{(1)}(t)$. Considering relations (3.7), we have

$$|x_i^{(2)}(t)|=\Big|x_i(t)-\frac{1}{h}\int\limits_t^{t+h}x_i(s)\,ds\Big|=|x_i(t)-x_i(\xi)|\le\omega(x_i,h),$$

where $t\in[-\tau,T]$.

If we now assume that $x_i(t) = x_i^{(1)}(t) + x_i^{(2)}(t)$ in the system (3.5), (3.6), then according to its linearity, the solution will be the sum of the functions that are solutions of the following systems:

$$
\begin{aligned}
\frac{\tau}{m}\frac{dz_{1i}^{(1)}(t)}{dt} + z_{1i}^{(1)}(t) &= x_i^{(1)}(t), \quad i = \overline{1,n},\\
\frac{\tau}{m}\frac{dz_{ji}^{(1)}(t)}{dt} + z_{ji}^{(1)}(t) &= z_{j-1,i}^{(1)}(t), \quad j = \overline{2,m}, \quad i = \overline{1,n},\\
z_{ji}^{(1)}(0) = y_{ji}(0), &\quad j = \overline{1,m}, \quad i = \overline{1,n},\\
\frac{\tau}{m}\frac{dz_{1i}^{(2)}(t)}{dt} + z_{1i}^{(2)}(t) &= x_i^{(2)}(t), \quad i = \overline{1,n},\\
\frac{\tau}{m}\frac{dz_{ji}^{(2)}(t)}{dt} + z_{ji}^{(2)}(t) &= z_{j-1,i}^{(2)}(t), \quad j = \overline{2,m}, \quad i = \overline{1,n},\\
z_j^{(2)}(0) = 0, &\quad j = \overline{1,m}, \quad i = \overline{1,n}.
\end{aligned}
$$

Thus, we get

$$
|z_{ji}(t) - y_{ji}(t)| = \left|z_{ji}^{(1)}(t) + z_{ji}^{(2)}(t) - y_{ji}^{(1)}(t) - y_{ji}^{(2)}(t)\right| \le |z_{ji}^{(1)}(t) - y_{ji}^{(1)}(t)| + |z_{ji}^{(2)}(t)| + |y_{ji}^{(2)}(t)|.
$$

It is obvious that

$$
|y_{ji}^{(2)}(t)| = \left|x_i^{(2)}\Big(t - \frac{j\tau}{m}\Big)\right| \le \omega(x_i, h).
$$

It is easy to show by mathematical induction that the same inequality holds for $|z_{ji}^{(2)}(t)|$.

To estimate the difference $|z_{ji}^{(1)}(t) - y_{ji}^{(1)}(t)|$, inequality (3.4) can be applied, since $x_i^{(1)}(t)$ satisfies the Lipschitz condition with some constant K_i. Therefore, we have the estimate

$$
|z_{ji}(t) - y_{ji}(t)| \le 2\Big(K_i\frac{\tau}{\sqrt{m}} + \omega(x_i, h)\Big).
$$

By setting $h = \frac{\tau}{\sqrt{m}}$, we get

$$
|z_{ji}(t) - y_{ji}(t)| \le 2\Big(K_i\frac{\tau}{\sqrt{m}} + \omega\Big(x_i, \frac{\tau}{\sqrt{m}}\Big)\Big). \tag{3.8}
$$

Adding inequalities (3.8), we obtain the estimate

$$
\sum_{i=1}^{n} |z_{ji}(t) - y_{ji}(t)| \le 2\sum_{i=1}^{n}\Big(\frac{\tau}{\sqrt{m}}K_i + \omega\Big(x_i, \frac{\tau}{\sqrt{m}}\Big)\Big) = \gamma\Big(\frac{\tau}{\sqrt{m}}\Big).
$$

Let us summarize the above considerations on the approximation of the delay element in the form of the following statement.

Theorem 3.1. *Let the input function $x(t)$ in system (3.2) be continuous for $t \in [-\tau, T]$. Then, for the solutions of the Cauchy problem (3.2), (3.3), the inequalities*

$$
\sum_{i=1}^{n} |z_{ji}(t) - y_{ji}(t)| \le \gamma\Big(\frac{\tau}{\sqrt{m}}\Big), \quad j = \overline{1,m}
$$

hold, where $\gamma(\delta)$ is a monotonously increasing function and $\lim_{\delta\to 0}\gamma(\delta) = 0$.

4 Analysis of the approximation scheme

Let us investigate the conditions for approximating the solutions of the initial problem (2.1) by the solutions of the Cauchy problem (2.3), (2.4).

We introduce the relation $z_j(t) = z_j^{(1)}(t) + z_j^{(2)}(t)$, where $z_j^{(1)}(t)$ and $z_j^{(2)}(t)$ are the solutions of the Cauchy problems

$$\begin{aligned} &\frac{\tau}{m}\frac{dz_1^{(1)}(t)}{dt} + z_1^{(1)}(t) = x(t), \\ &\frac{\tau}{m}\frac{dz_j^{(1)}(t)}{dt} + z_1^{(1)}(t) = z_{j-1}(t), \quad j=\overline{2,m}, \quad t\in[0,T], \\ &\qquad z_j^{(1)}(0) = x\Big(-\frac{j\tau}{m}\Big), \quad j=\overline{1,m}; \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\frac{\tau}{m}\frac{dz_1^{(2)}(t)}{dt} + z_2^{(2)}(t) = z_0(t) - x(t), \\ &\frac{\tau}{m}\frac{dz_j^{(2)}(t)}{dt} + z_j^{(2)}(t) = z_{j-1}^{(2)}(t), \quad j=\overline{2,m}, \quad t\in[0,T], \\ &\qquad z_j^{(2)}(0) = 0, \quad j=\overline{1,m}. \end{aligned} \tag{4.2}$$

Taking into account the structure of linear systems (4.1), (4.2) and the representation of the functions $z_j(t)$, $j=\overline{1,m}$, we estimate the difference $z_j(t) - x(t-\frac{j\tau}{m})$, $j=\overline{1,m}$.

We have the relation

$$\Big\|z_j(t) - x\Big(t-\frac{j\tau}{m}\Big)\Big\| = \Big\|z_j^{(1)}(t) + z_j^{(2)}(t) - x\Big(t-\frac{j\tau}{m}\Big)\Big\| \le \Big\|z_j^{(1)}(t) - x\Big(t-\frac{j\tau}{m}\Big)\Big\| + \|z_j^{(2)}(t)\|.$$

Considering the representation $z_j(t) = (z_{j1}(t),\dots,z_{jn}(t))$, $x(t) = (x_1(t),\dots,x_n(t))$, we present $z_{ji}(t)$, $j=\overline{1,m}$, $i=\overline{1,n}$, in the form of a sum $z_{ji}(t) = z_{ji}^{(1)}(t) + z_{ji}^{(2)}(t)$, where $z_{ji}^{(1)}(t)$ and $z_{ji}^{(2)}(t)$ are the solutions of the Cauchy problems

$$\begin{aligned} &\frac{\tau}{m}\frac{dz_{j1}^{(1)}(t)}{dt} + z_{j1}^{(1)}(t) = x_i(t), \\ &\frac{\tau}{m}\frac{dz_{ji}^{(1)}(t)}{dt} + z_{ji}^{(1)}(t) = z_{j-1,i}^{(1)}(t), \quad j=\overline{2,m}, \quad i=\overline{1,n}, \\ &\qquad z_{ji}^{(1)}(0) = x_i\Big(-\frac{j\tau}{m}\Big), \quad j=\overline{1,m}, \quad i=\overline{1,n}; \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\frac{\tau}{m}\frac{dz_{j1}^{(2)}(t)}{dt} + z_{j1}^{(2)}(t) = z_{01}(t) - x_i(t), \\ &\frac{\tau}{m}\frac{dz_{ji}^{(2)}(t)}{dt} + z_{ji}^{(2)}(t) = z_{j-1,i}^{(2)}(t), \quad j=\overline{2,m}, \quad i=\overline{1,n}, \\ &\qquad z_{ji}^{(2)}(0) = 0, \quad j=\overline{1,m}, \quad i=\overline{1,n}. \end{aligned} \tag{4.4}$$

Denote

$$N_j(t) = \max_{0\le s\le t}\sum_{i=1}^n \Big|x_i\Big(s-\frac{\tau j}{m}\Big) - z_{ij}(s)\Big|, \quad j=\overline{0,m}, \quad t\in[0,T].$$

Taking into account the structure of systems (4.3), (4.4), for the difference $\sum_{i=1}^n |x_i(t-\frac{\tau j}{m}) - z_{ij}(t)|$, we have the inequality

$$\sum_{i=1}^n \Big|x_i\Big(t-\frac{j\tau}{m}\Big) - z_{ij}(t)\Big| \le \sum_{i=1}^n \Big|x_i\Big(t-\frac{j\tau}{m}\Big) - z_{ji}^{(1)}(t)\Big| + \sum_{i=1}^n |z_{ji}^{(2)}(t)|. \tag{4.5}$$

It is easy to show by the method of mathematical induction that for the second term on the right-hand side of inequality (4.5) the following estimate holds:

$$\sum_{i=1}^n |z_{ji}^{(2)}(t)| \le N_0(t), \quad j=\overline{1,m}, \quad t\in[0,T]. \tag{4.6}$$

For the initial problem (2.1), we have that $x_i(t) \in C[-\tau, T]$, $i = \overline{1,n}$, therefore, to estimate the difference $|x_i(t - \frac{j\tau}{m}) - z_{ji}^{(1)}(t)|$, we can apply Theorem 3.1. Then we obtain the estimate

$$\sum_{i=1}^{n} \left| x_i\Big(t - \frac{j\tau}{m}\Big) - z_{ji}^{(1)}(t) \right| \le \delta\Big(\frac{\tau}{\sqrt{m}}\Big), \quad \lim_{s \to 0} \delta(s) = 0. \tag{4.7}$$

Inequality (4.7) holds for all $t \in [0, T]$, therefore, taking into account inequalities (4.6), (4.7), from relation eq27 we obtain $N_j(t) \le \delta(\frac{\tau}{\sqrt{m}}) + N_0(t)$, $j = \overline{1,m}$.

We denote

$$K_A = \max_{i=\overline{0,p}} \max_t \|A_i(t)\|, \quad K_D = \max_{t,\theta} \|D(t,\theta)\|, \quad \omega_1\Big(\frac{\tau}{m}\Big) = n \cdot \max_{i,j} \omega\Big(d_{ij}, \frac{\tau}{m}\Big),$$

$\omega(d_{ij}, \frac{\tau}{m})$ is the continuity modulus of the function $d_{ij}(t,\theta)$, $t \in [0, T]$, $\theta \in [-\tau, 0]$, $i, j = \overline{1,n}$.

To obtain estimates of the solutions of system (2.3), (2.4), let $\sum\limits_{i=1}^{n} |z_{ij}(0)| \le \gamma$, $j = \overline{0,m}$. We denote

$$M(t) = \max_{0 \le s \le t} \Big[\gamma, \sum_{i=1}^{n} |z_{0i}(s)|\Big].$$

From the vector equation

$$\frac{dz_1(t)}{dt} = \frac{m}{\tau}\,(z_0(t) - z_1(t)),$$

we obtain

$$z_{1i}(t) = z_{1i}(0)\exp\Big(-\frac{m}{\tau}t\Big) + \frac{m}{\tau}\int_0^t z_{0i}(s)\exp\Big(-\frac{m}{\tau}(t-s)\Big)\,ds.$$

Therefore,

$$\sum_{i=1}^{n} |z_{1i}(t)| \le M(t)\exp\Big(-\frac{m}{\tau}t\Big) + \frac{m}{\tau}\int_0^t M(s)\exp\Big(-\frac{m}{\tau}(t-s)\Big)\,ds$$

$$\le M(t)\Big[\exp\Big(-\frac{m}{\tau}t\Big) + \frac{m}{\tau}\int_0^t \exp\Big(-\frac{m}{\tau}(t-s)\Big)\,ds\Big] = M(t).$$

Continuing in this way, we get $\sum\limits_{i=1}^{n} |z_{ji}(t)| \le M(t)$, $j = \overline{1,m}$.

Let us write equations (2.3), (2.4) in the equivalent integral form

$$x(t) = \varphi(0) + \int_0^t \sum_{i=0}^{p} A_i(s)x(s-\tau_i)\,ds + \int_0^t \sum_{j=0}^{m-1} \int_{-\tau+\frac{j\tau}{m}}^{-\tau+(j+1)\frac{\tau}{m}} D(s,\theta)x(s+\theta)\,d\theta\,ds, \tag{4.8}$$

$$z_0(t) = \varphi(0) + \int_0^t \sum_{i=0}^{p} A_i(s)z_{l_i}(s)\,ds + \int_0^t \sum_{j=0}^{m-1} \int_{-\tau+\frac{j\tau}{m}}^{-\tau+(j+1)\frac{\tau}{m}} D\Big(s, -\frac{\tau}{m}(m-j)\Big)z_{m-j}(s)\,d\theta\,ds. \tag{4.9}$$

Taking into account inequality (4.9), we obtain

$$|z_{0i}(t)| \le |z_{0i}(0)| + (p+1)K_A\int_0^t M(s)\,ds + \tau K_D\int_0^t M(s)\,ds = |z_{0i}(0)| + ((p+1)K_A + \tau K_D)\int_0^t M(s)\,ds.$$

Now, we have

$$\sum_{i=1}^{n}|z_{i0}(t)| \le \sum_{i=1}^{n}|z_{i0}(0)| + n((p+1)K_A+\tau K_D)\int\limits_0^t M(s)\,ds.$$

Hence, $M(t)\le \gamma + n((p+1)K_A+\tau K_D)\int\limits_0^t M(s)\,ds$. Using the Gronwall–Bellman lemma, we obtain $M(t)\le \gamma e^{n((p+1)K_A+\tau K_D)T}=K_2$. From equations (4.8), (4.9), we find

$$\begin{aligned}
&\|x(t)-z_0(t)\| \le \sum_{i=0}^{p} K_A\int\limits_0^t \Big(N_0(s)+\delta_1\Big(\frac{\tau}{m}\Big)\Big)\,ds\\
&\quad+\int\limits_0^t\sum_{j=0}^{m-1}\int\limits_{-\tau+\frac{j\tau}{m}}^{-\tau+(j+1)\frac{\tau}{m}}\Big\|D(s,\theta)(x(s+\theta)-z_{m-j}(s))+\Big(D(s,\theta)-D\Big(s,\frac{\tau}{m}(m-j)\Big)\Big)z_{m-j}(s)\Big\|\,d\theta\,ds\\
&\le \sum_{i=0}^{p} K_A\int\limits_0^t \Big(N_0(s)+\delta_1\Big(\frac{\tau}{m}\Big)\Big)\,ds+\int\limits_0^t\sum_{j=0}^{m-1}\int\limits_{-\tau+\frac{j\tau}{m}}^{-\tau+(j+1)\frac{\tau}{m}}\Big(K_D\Big(\delta_1\Big(\frac{\tau}{m}\Big)+N_0(s)\Big)+K_2\omega_1\Big(\frac{\tau}{m}\Big)\Big)\,d\theta\,ds\\
&\le K_A(p+1)T\delta_1\Big(\frac{\tau}{m}\Big)+(p+1)K_A\int\limits_0^t N_0(s)\,ds+\tau T\Big(K_D\delta_1\Big(\frac{\tau}{m}\Big)+K_2\omega_1\Big(\frac{\tau}{m}\Big)\Big)+\tau K_D\int\limits_0^t N_0(s)\,ds\\
&\le (K_A(p+1)+\tau K_D)T\delta_1\Big(\frac{\tau}{m}\Big)+K_2\tau T\omega_1\Big(\frac{\tau}{m}\Big)+((p+1)K_A+\tau K_D)\int\limits_0^t N_0(s)\,ds,\quad t\in[0,T].
\end{aligned}$$

Using the Gronwall–Bellman lemma, we get

$$N_0(t)\le \Big[(K_A(p+1)+\tau K_D)\delta_1\Big(\frac{\tau}{m}\Big)+\tau K_2\omega_1\Big(\frac{\tau}{m}\Big)\Big]Te^{T[(p+1)K_A+\tau K_D]}. \tag{4.10}$$

Since

$$\lim_{m\to\infty}\delta_1\Big(\frac{\tau}{m}\Big)=0,\quad \lim_{m\to\infty}\omega_1\Big(\frac{\tau}{m}\Big)=0,$$

it follows from relation (4.10) that the solutions of the Cauchy problem (2.3), (2.4) approximate the solutions of the original problem (2.1) as $m\to\infty$.

Remark. Similarly, we can prove that the solutions of the Cauchy problem (2.5), (2.6) approximate the solutions of the original problem (2.1) as $m\to\infty$.

Let us formulate the obtained result in the form of a theorem.

Theorem 4.1. *If $A_i(t)$, $i=\overline{0,p}$, $D(t,\theta)$ are continuous matrix functions for $t\in[0,T]$, $\theta\in[-\tau,0]$, then the solutions of the Cauchy problems* (2.3), (2.4) *and* (2.5), (2.6) *approximate the solutions of the original problem* (2.1) *as $m\to\infty$.*

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Authors' addresses:

Ihor Cherevko
Yuriy Fedkovych Chernivtsi National University, 2 Kotsyubynsky Str., Chernivtsi 58012, Ukraine
E-mail: i.cherevko@chnu.edu.ua

Andrii Dorosh
Yuriy Fedkovych Chernivtsi National University, 2 Kotsyubynsky Str., Chernivtsi 58012, Ukraine
E-mail: a.dorosh@chnu.edu.ua

Svitlana Ilika
Yuriy Fedkovych Chernivtsi National University, 2 Kotsyubynsky Str., Chernivtsi 58012, Ukraine
E-mail: s.ilika@chnu.edu.ua

Dmytro Shkilniuk
Yuriy Fedkovych Chernivtsi National University, 2 Kotsyubynsky Str., Chernivtsi 58012, Ukraine
E-mail: dimonshk@gmail.com