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WELL-POSEDNESS AND ENERGY DECAY OF A THIN PLATE WITH DYNAMICS IN THERMOELASTICITY AND MASS DIFFUSION

Abstract. In this note, we prove the global existence and the asymptotic behavior of solutions for a thermoelastic plate with mass diffusion. By assuming certain conditions on the parameters of the model which present sufficient conditions for the stability result, the stabilization under control of the Lyapunov functional is obtained only for a localized initial conditions in the ball of radius R.

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რეზიუმე. ნაშრომში დამტკიცებულია ამონახსნების გლობალური არსებობა და ასიმპტოტური ყოფაქცევა თერმოდრეკადი ფირფიტისთვის მასის დიფუზიით. მოდელის პარამეტრებზე გარკვეული პირობების დაშვებით, რომლებიც წარმოადგენს სტაბილურობის საკმარის პირობებს, ლიაპუნოვის ფუნქციონალის კონტროლის ქვეშ მყოფი სტაბილიზაცია მიღებულია მხოლოდ R რადიუსის მქონე ბირთვში ლოკალიზებული საწყისი პირობებისთვის.

1 Introduction

Stability is one of the most important characteristics around which theoretical and experimental studies of different physical models revolve. It can be accommodated by PDEs or coupled systems of PDEs. These systems often express change and development with respect to space and time, where they can model many physical phenomena, for example, the vibration of beams, the flow and velocity of fluids, diffusion of chemicals, etc. As for the previous models, we may find that stability is the basis for the mathematical and physicist researcher to develop other models as well as obtain reliable results in reality. In this paper, we study a nonlinear mechanical model of the van Káramán type coupled with thermal effects and take the mass diffusion into account.

Menzala and Zuazua [12] have investigated the asymptotic behavior of solutions for the van Káramán system and proved the exponential stability of the system during temperature coupling. We can point out that the van Káramán system has been considered in many studies, among which the noteworthy are the works [2, 3]. More precisely, the researchers have shown that the presence of thermal effects leads the system to stability, and this result is identical to the van Káramán equation. With regard to heat transfer, several models have been developed, for example, the transfer according to the Fourier law, Green–Naghdi law, Cattaneo law, Gurtin–Pipkin law, etc. These models have proven their effectiveness in reality after being subjected to experimental studies that exist to this day. Recently, the heat flow under different laws have attracted the interest of many researchers whose main goal was to show the behavior of solutions for mechanical systems in the presence of different thermal effects, as a special case of the van Káramán system, we can refer to [5, 7, 8, 10] and the references therein.

Favini et al. [6], using the non-linear boundary dissipations, proved the global existence, uniqueness and regularity of the van Káramán system. As in most studies on the van Káramán system, rotational inertia was not taken into account and it was also taken into account that thermal coupling was partial, i.e., with one equation of the mechanical system, it was a challenge for us to know the behavior of the solutions in the opposite case to that. In [4], the thermodiffusion van Káramán system with time delay was taken into account, as the study proved the exponential stability of energy. The effects of thermal diffusion were initially suggested by Aouadi et al. [1], where the authors studied the Timoshenko beam, which expressed the linear transversal displacement and the shear angle of the beam, and they have shown the existence, stability and numerical results. In our work, we considered the existence of strong thermal diffusion and the aforementioned properties and show the global existence as well as the exponential stability of the solutions. In the following section, we present the derivation of our studied model based on the work of Lagnese and Leugering [9].

Derivation of the model

We suppose that the beam occupies the region

$$\Big\{(x_1, x_2, x_3): \quad 0 \le x_1 \le L, \quad -1 \le x_2 \le 1, \quad -\frac{h}{2} \le x_3 \le \frac{h}{2}\Big\},$$

its centerline is defined by $0 \le x_1 \le L$, $x_2 = x_3 = 0$, and the cross-sections are

$$A(x_1) = \Big\{(x_1, x_2, x_3): \quad x_1 = x_1, \quad -1 \le x_2 \le 1, \quad -\frac{h}{2} \le x_3 \le \frac{h}{2}\Big\}.$$

By $r(x_1, t)$ we denote the position vector at time t of the particle which occupies position $(x_1, 0, 0)$ on the centerline in the reference configuration $r(x_1, t) - (x_1, 0, 0)$. The centerline is constrained to move in the e_1e_3-plane as follows:

$$r(x_1, t) = (u(x_1, t) + x_1)e_1 + w(x_1, t)e_3,$$

where the functions u and w are, respectively, the longitudinal and the transversal displacements of the point $(x_1, 0, 0)$. Now, if the deformation is taken into account, the point x_1 on the centerline is mapped onto a point p in the e_1e_3-plane whose abscissa is $x_1 + u$ and ordinate is w, then this deformation causes an axial stretching $s(x_1,t)$ given by

$$s(x_1,t)=\int_0^{x_1}\left[(1+u_{x_1}(\xi,t))^2+(w_{x_1}(\xi,t))^2\right]^{1/2}d\xi - x_1. \tag{1.1}$$

The body is subjected to an unknown heat distribution τ and a chemical potential $\widetilde{\tau}$ that vanishes at the boundary.

Any set of forces acting on the particular cross-section located at x_1 in the undeformed state can be replaced by the torque T, the resultant force R, the thermal strain resultant Q and the chemical strain resultant S such that

$$\begin{aligned} T&=T_1e_1+M_2e_2+M_3e_3,\\ R&=P_1e_1+V_2e_2+V_3e_3,\\ Q&=\theta_1e_1+\theta_2e_2+\theta_3e_3,\\ S&=\vartheta_1e_1+\vartheta_2e_2+\vartheta_3e_3, \end{aligned} \tag{1.2}$$

where T_i is an axial torque, M_i is a bending moment about e_i, V_i are the shear components of R, θ_i are the heat dissipations and ϑ_i are the mass diffusion. Now, taking $V_2=V_3=0$ and $T_1=M_3=0$ in the previous equations (1.2), we can get

$$\begin{aligned} &P_1=EAs_{x_1}(x_1,t), \quad M_2=-EIw_{x_1x_1}(x_1,t),\\ &\theta_1=\frac{1}{h}\int_{-h/2}^{h/2}\tau\,dx_3, \quad \theta_2=\frac{12}{h^3}\int_{-h/2}^{h/2}x_3\tau\,dx_3,\\ &\vartheta_1=\frac{1}{h}\int_{-h/2}^{h/2}\widetilde{\tau}\,dx_3, \quad \vartheta_2=\frac{12}{h^3}\int_{-h/2}^{h/2}x_3\widetilde{\tau}\,dx_3, \end{aligned}$$

where A is the area of the cross-section, I is its moment of inertia with respect to the x_2-axis and Young's modulus E. EI is known as the flexural rigidity. Therefore, the strain energy is

$$U=\int_0^L\frac{P_1^2}{2EA}\,dx_1+\int_0^L\frac{M_2^2}{2EI}\,dx_1=\frac{1}{2}\int_0^L EAs_{x_1}^2\,dx_1+\frac{1}{2}\int_0^L EIw_{x_1x_1}^2\,dx_1.$$

We define the energy coming from the heat conduction by

$$\Theta=\int_0^L -\gamma_1\theta_1u_{x_1}\,dx_1-\int_0^L\gamma_3\theta_2w_{x_1x_1}\,dx_1,$$

where γ_1 and γ_3 are the coefficients of thermal expansions. Then the energy coming from the mass diffusion is

$$\Sigma=\int_0^L -\gamma_2\vartheta_1u_{x_1}\,dx_1-\int_0^L\gamma_4\vartheta_2w_{x_1x_1}\,dx_1,$$

where γ_2 and γ_4 are the coefficients of mass diffusion expansions. From (1.1) we have

$$s_{x_1}(x_1,t)=u_{x_1}(x_1,t)+\frac{1}{2}\,(w_{x_1}(x_1,t))^2+\frac{1}{2}\,(u_{x_1}(x_1,t))^2. \tag{1.3}$$

Taking into account the first two terms in (1.3), the strain energy takes the form

$$U=\frac{EA}{2}\int_0^L\left[u_{x_1}+\frac{1}{2}\,(w_{x_1})^2\right]^2dx_1+\frac{EI}{2}\int_0^L(w_{x_1x_1})^2\,dx_1,$$

and the kinetic energy is defined by

$$K = \frac{\rho I}{2} \int_0^L (w_{tx_1})^2 \, dx_1 + \frac{\rho A}{2} \int_0^L (u_t)^2 + (w_t)^2 \, dx_1,$$

where ρ is the mass density per unit volume of the beam.

Now, it is time to define the Lagrangian density as follows:

$$\widehat{L}(u_{x_1}, w_{x_1}, w_{x_1x_1}, u_t, w_t, w_{tx_1}).$$

Following Hamilton's principle for contituous systems, we have to introduce variations of the field quantities u and w. As a necessary condition for the Lagrangian $\mathcal{L}$ to be stationary at u, w, the Gateaux derivative $\delta\mathcal{L}$ of

$$\mathcal{L} = \int_0^T \int_0^L \widehat{L}(u_{x_1}, w_{x_1}, w_{x_1x_1}, u_t, w_t, w_{tx_1}) \, dx_1 \, dt$$

with respect to these variations must be zero. Hence the result of calculations is the following system with boundary conditions:

$$\begin{cases} \rho A u_{tt} - EA\Big(u_{x_1} + \frac{1}{2}(w_{x_1})^2\Big)_{x_1} - \gamma_1 \theta_{1x_1} - \gamma_2 \vartheta_{1x_1} = 0, \\ \rho A w_{tt} - \rho I w_{ttx_1x_1} - EA\Big[\Big(u_{x_1} + \frac{1}{2}(w_{x_1})^2\Big) w_{x_1}\Big]_{x_1} \\ \qquad\qquad + EI w_{x_1x_1x_1x_1} + \gamma_3 \theta_{2x_1x_1} + \gamma_4 \vartheta_{2x_1x_1} = 0, \\ u(0,t) = w(0,t) = w_{x_1}(0,t) = 0. \end{cases} \tag{1.4}$$

In order to simplify notation, we introduce the following changes: $\gamma^2 = I/A$ and $t \to t\sqrt{\rho/E}$ and $x_1 \to x$. In the next sections of the paper, the use will be made of the following notations:

$$\begin{gathered} \Big(u_x + \frac{1}{2}(w_x)^2\Big) = \Psi(u,w), \quad \Phi_\gamma = (I - \gamma^2 \partial_{xx}), \\ \partial_{xxxx} = \partial_x^4, \quad \partial_{xx} = \partial_x^2. \end{gathered} \tag{1.5}$$

System (1.4) takes the form

$$\begin{cases} u_{tt} - \Big(u_x + \frac{1}{2}(w_x)^2\Big)_x - \gamma_1 \theta_{1x} - \gamma_2 \vartheta_{1x} = 0, \\ w_{tt} + \gamma^2 w_{ttxx} - \Big[\Big(u_x + \frac{1}{2}(w_x)^2\Big) w_x\Big]_x + \gamma^2 w_{xxxx} + \gamma_3 \theta_{2xx} + \gamma_4 \vartheta_{2xx} = 0, \\ u(x,t) = w(x,t) = w_x(x,t) = 0. \end{cases} \tag{1.6}$$

The temperature is governed by the following system of equations:

$$\begin{cases} c_1 \theta_{1t} + d_1 \vartheta_{1t} - \gamma_1 u_{tx} = -q_{1x}, \\ c_2 \theta_{2t} + d_2 \vartheta_{2t} - \gamma_3 w_{txx} = -q_{2x}, \\ \theta_1(x,t) = \theta_2(x,t) = 0, \end{cases} \tag{1.7}$$

where c_i is a parameter from the thermoelasticity theory, d_i is a measure of the thermodiffusion effect and q_i is the heat flux that will be considered here under Fourier's law, i.e.,

$$q_i = -\kappa_i \theta_{ix}, \quad \kappa_i\text{: heat conductivity coefficient.} \tag{1.8}$$

The chemical potential is governed by the following system of equations:

$$\begin{cases} d_1 \theta_{1t} + r_1 \vartheta_{1t} - \gamma_2 u_{tx} = -\eta_{1x}, \\ d_2 \theta_{2t} + r_2 \vartheta_{2t} - \gamma_4 w_{txx} = -\eta_{2x}, \\ \vartheta_1(x,t) = \vartheta_2(x,t) = 0, \end{cases} \tag{1.9}$$

where r_i is a measure of the diffusive effect and

$$-\eta_{ix} = C_{it}$$

such that C is the concentration of the diffusive material in the elastic body and η_i is the mass diffusion flux that will be considered here under Fick's law, i.e.,

$$\eta_i = -\nu_i \vartheta_x, \quad \nu_i\text{: mass diffusion conductivity coefficient.} \tag{1.10}$$

Along this paper, the index i takes the value 1 or 2, respectively. By using notations (1.5) and rearranging systems (1.6)–(1.9) with laws (1.8) and (1.10) we obtain the following system:

$$\begin{cases} \partial_t^2 u - \partial_x \Psi(u,w) - \gamma_1 \partial_x \theta_1 - \gamma_2 \partial_x \vartheta_1 = 0, \\ \Phi_\gamma \partial_t^2 w - \partial_x(\Psi(u,w)\partial_x w) + \gamma^2 \partial_x^4 w + \gamma_3 \partial_x^2 \theta_2 + \gamma_4 \partial_x^2 \vartheta_2 = 0, \\ c_1 \partial_t \theta_1 - \kappa_1 \partial_x^2 \theta_1 + d_1 \partial_t \vartheta_1 - \gamma_1 \partial_{tx} u = 0, \\ d_1 \partial_t \theta_1 - \nu_1 \partial_x^2 \vartheta_1 + r_1 \partial_t \vartheta_1 - \gamma_2 \partial_{tx} u = 0, \\ c_2 \partial_t \theta_2 - \kappa_2 \partial_x^2 \theta_2 + d_2 \partial_t \vartheta_2 - \gamma_3 \partial_x^2 \partial_t w = 0, \\ d_2 \partial_t \theta_2 - \nu_2 \partial_x^2 \vartheta_2 + r_2 \partial_t \vartheta_2 - \gamma_4 \partial_x^2 \partial_t w = 0. \end{cases} \tag{1.11}$$

For $(x,t) \in (0,L) \times \mathbb{R}^+$, the system is associated with the boundary conditions

$$\begin{aligned} &u_x(t,x) = w(t,x) = w_x(t,x) = \theta_1(t,x), \;\; t \in \mathbb{R}^+, \;\; x = 0 \text{ or } x = L. \\ &\theta_2(t,x) = \vartheta_1(t,x) = \vartheta_2(t,x) = 0, \;\; t \in \mathbb{R}^+, \;\; x = 0 \text{ or } x = L, \end{aligned} \tag{1.12}$$

and with the initial conditions

$$\begin{cases} u(x,0) = u^0(x), & u_t(x,0) = u^1(x), \\ \theta_1(x,0) = \theta_1^0(x), & \vartheta_1(x,0) = \vartheta_1^0(x), \\ w(x,0) = w^0(x), & w_t(x,0) = w^1(x), \\ \theta_2(x,0) = \theta_2^0(x), & \vartheta_2(x,0) = \vartheta_2^0(x). \end{cases} \tag{1.13}$$

1.1 Functional space and assumption

Initial conditions (1.13) are considered in the associated energy (phase) space:

$$\mathcal{H} = H^1(0,L) \cap L_*^2(0,L) \times (L^2(0,L))^3 \times H_0^2(0,L) \times H_0^1(0,L) \times (L^2(0,L))^2, \tag{1.14}$$

where

$$L_*^2(0,L) = \left\{ f \in L^2(0,L) : \int_0^L f(x)\,dx = 0 \right\}.$$

The symbols $\langle \cdot, \cdot \rangle$, $\| \cdot \|$ and $| \cdot |$ denote the L^2-inner product, L^2-norm and L^∞-norm, respectively.

For $z = (u, v, \theta_1, \vartheta_1, w, y, \theta_2, \vartheta_2)^T$ and $\widetilde{z} = (\widetilde{u}, \widetilde{v}, \widetilde{\theta}_1, \widetilde{\vartheta}_1, \widetilde{w}, \widetilde{y}, \widetilde{\theta}_2, \widetilde{\vartheta}_2)^T$, the Hilbert space (1.14) is endowed with inner product as follows:

$$\begin{aligned} \langle z, \widetilde{z} \rangle_{\mathcal{H}} = \langle v, \overline{\widetilde{v}} \rangle + \left\langle \Phi_\gamma^{\frac{1}{2}} y, \Phi_\gamma^{\frac{1}{2}} \overline{\widetilde{y}} \right\rangle + \langle \partial_x u, \partial_x \overline{\widetilde{u}} \rangle + \gamma^2 \langle \partial_x^2 w, \partial_x^2 \overline{\widetilde{w}} \rangle \\ + \left\langle \Lambda_1 \begin{pmatrix} \theta_1 \\ \vartheta_1 \end{pmatrix}, \begin{pmatrix} \overline{\widetilde{\theta}}_1 \\ \overline{\widetilde{\vartheta}}_1 \end{pmatrix} \right\rangle + \left\langle \Lambda_2 \begin{pmatrix} \theta_2 \\ \vartheta_2 \end{pmatrix}, \begin{pmatrix} \overline{\widetilde{\theta}}_2 \\ \overline{\widetilde{\vartheta}}_2 \end{pmatrix} \right\rangle, \end{aligned}$$

with the matrix

$$\Lambda_i = \begin{pmatrix} c_i & d_i \\ d_i & r_i \end{pmatrix}.$$

The Hilbert space (1.14) is equipped with the norm

$$\|z\|_{\mathcal{H}}^2 = \|v\|^2 + \|\partial_x u\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 + \gamma^2\|\partial_x^2 w\|^2 + A(t) \tag{1.15}$$

such that

$$A(t) := c_i\|\theta_i\|^2 + 2d_i\Re\langle\theta_i, \vartheta_i\rangle + r_i\|\vartheta_i\|^2.$$

Assume that the matrices Λ_i are positive definite, that is,

$$\delta_i := \det(\Lambda_i) > 0 \Longrightarrow A(t) > 0. \tag{1.16}$$

Let ξ_i be a number chosen such that $d_i/c_i < \xi_i < r_i/d_i$, then by the Cauchy–Schwarz and Young inequalities

$$2d_i\Re\langle\theta_i, \vartheta_i\rangle_2 \le \frac{d_i}{\xi_i}\|\theta_i\|^2 + d_i\xi_i\|\vartheta_i\|^2. \tag{1.17}$$

Using (1.15) and (1.17), we get

$$\|z\|_{\mathcal{H}}^2 \le \|v\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 + \|\partial_x u\|^2 + \gamma^2\|\partial_x^2 w\|^2 + \alpha\|\theta_i\|^2 + \beta\|\vartheta_i\|^2,$$

where

$$\alpha = c_i + \frac{d_i}{\xi_i} \ \text{ and } \ \beta = r_i + d_i\xi_i.$$

The following remark is needed for the next sections of this paper.

Remark 1.1. Recalling (1.5), we have

$$\begin{aligned}
(-\partial_x^2)^{-1} : L^2(0,L) \to H^2(0,L) \cap H_0^1(0,L) &\ \text{ is a Bounded operator,}\\
\Phi_\gamma^{-1} : L^2(0,L) \to H^2(0,L) \cap H_0^1(0,L) &\ \text{ is a Bounded operator,}\\
\Phi_\gamma^{-1}\partial_x : L^2(0,L) \to H_0^1(0,L) &\ \text{ is a Bounded operator,}\\
\langle \Phi_\gamma s, \widetilde{s}\rangle = \langle s, \Phi_\gamma^* \widetilde{s}\rangle = \langle s, \Phi_\gamma \widetilde{s}\rangle &\ \text{ is a Self-adjoint operator.}
\end{aligned} \tag{1.18}$$

Continuous Energy

Set $\partial_t u = v$, $\partial_t w = y$. Multiplying equations of system (1.11) by the functions $(v, y, \theta_1, \vartheta_1, \theta_2, \vartheta_2)$, respectively, and integrating over $(0, L)$, we obtain the following energy functional:

$$\mathcal{E}'(t) = \frac{1}{2}\left[\|v\|^2 + \gamma^2\|\partial_x^2 w\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 + \|\Psi\|^2 + A(t)\right]' = -\kappa_i\|\partial_x\theta_i\|^2 - \nu_i\|\partial_x\vartheta_i\|^2. \tag{1.19}$$

On the one hand, from (1.15) we have

$$\begin{aligned}
\|z\|_{\mathcal{H}}^2 &= \|v\|^2 + \left\|\partial_x u + \frac{1}{2}(\partial_x w)^2 - \frac{1}{2}(\partial_x w)^2\right\|^2 + \gamma^2\|\partial_x^2 w\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 + A(t)\\
&\le \|v\|^2 + \gamma^2\|\partial_x^2 w\|^2 + 2\left\|\left(\frac{1}{2}\,\partial_x w\right)^2\right\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 + 2\|\Psi\|^2 + A(t)\\
&\le 2\sqrt{\mathcal{E}(t)}\,\mathcal{E}(t).
\end{aligned} \tag{1.20}$$

In contrast, we have

$$\mathcal{E}(t) \le \|z\|_{\mathcal{H}}^2\|z\|_{\mathcal{H}}.$$

Hence there exist the constants ξ_3 and ξ_4 such that

$$\xi_3\|z\|_{\mathcal{H}}^2 \le \mathcal{E}(t) \le \xi_4\|z\|_{\mathcal{H}}^2. \tag{1.21}$$

Discret Energy

Multiplying (1.11) by the test functions $\chi_j \in H_0^1(0,L)$ for $j = 1,\dots,6$, we obtain the following weak form:

$$\begin{cases} \langle \partial_t v, \chi_1 \rangle + \big\langle \Psi(u,w), \partial_x \chi_1 \big\rangle + \gamma_1 \langle \theta_1, \partial_x \chi_1 \rangle + \gamma_2 \langle \vartheta_1, \partial_x \chi_1 \rangle = 0, \\ \langle \Phi_\gamma \partial_t y, \chi_2 \rangle + \big\langle (\Psi(u,w)\partial_x w), \partial_x \chi_2 \big\rangle + \gamma^2 \langle \partial_x^4 w, \chi_2 \rangle + \gamma_3 \langle \partial_x^2 \theta_2, \chi_2 \rangle + \gamma_4 \langle \partial_x^2 \vartheta_2, \chi_2 \rangle = 0, \\ c_1 \langle \partial_t \theta_1, \chi_3 \rangle + \kappa_1 \langle \partial_x \theta_1, \partial_x \chi_3 \rangle + d_1 \langle \partial_t \vartheta_1, \chi_3 \rangle - \gamma_1 \langle \partial_x v, \chi_3 \rangle = 0, \\ d_1 \langle \partial_t \theta_1, \chi_4 \rangle + \nu_1 \langle \partial_x \vartheta_1, \partial_x \chi_4 \rangle + r_1 \langle \partial_t \vartheta_1, \chi_4 \rangle - \gamma_2 \langle \partial_x v, \chi_4 \rangle = 0, \\ c_2 \langle \partial_t \theta_2, \chi_5 \rangle + \kappa_2 \langle \partial_x \theta_2, \partial_x \chi_5 \rangle + d_2 \langle \partial_t \vartheta_2, \chi_5 \rangle + \gamma_3 \langle \partial_x y, \partial_x \chi_5 \rangle = 0, \\ d_2 \langle \partial_t \theta_2, \chi_6 \rangle + \nu_2 \langle \partial_x \vartheta_2, \partial_x \chi_6 \rangle + r_2 \langle \partial_t \vartheta_2, \chi_6 \rangle + \gamma_4 \langle \partial_x y, \partial_x \chi_6 \rangle = 0. \end{cases} \tag{1.22}$$

Let us partition the interval $(0,L)$ into subintervals $\mathcal{I}_j = (x_{j-1}, x_j)$ of length $h = 1/s$ with $0 = x_0 < \dots < x_s = L$ and define

$$S_0^h = \big\{ \eta \in H_0^1(0,L) : \quad \eta \in C([0,L]), \quad \eta|_{I_j} \ \text{ is a linear polynomial} \big\}.$$

For a given final time T and a positive integer N, let $\Delta T = T/N$ be the time step and $t_n = n\Delta t$, $n = 0,\dots,N$. The finite element method for (1.22) with the boundary conditions is to find $v_h^n, y_h^n, \theta_{1h}^n, \vartheta_{1h}^n, \theta_{2h}^n, \vartheta_{2h}^n$ such that, for all $\chi_1,\dots,\chi_6 \in S_0^h$,

$$\begin{cases} \dfrac{1}{\Delta t} \langle v_h^n - v_h^{n-1}, \chi_{1h} \rangle + \big\langle \Psi(u_h^n, w_h^n), \partial_x \chi_{1h} \big\rangle \\ \qquad + \gamma_1 \langle \theta_{1h}^n, \partial_x \chi_{1h} \rangle + \gamma_2 \langle \vartheta_{1h}^n, \partial_x \chi_{1h} \rangle = 0, \\ \dfrac{1}{\Delta t} \langle \Phi_\gamma^{\frac12} y_h^n - \Phi_\gamma^{\frac12} y_h^{n-1}, \Phi_\gamma^{\frac12} \chi_2 \rangle + \langle (\Psi(u_h^n, w_h^n) \partial_x w_h^n), \partial_x \chi_{2h} \rangle \\ \qquad + \gamma^2 \langle \partial_x^4 w_h^n, \chi_{2h} \rangle + \gamma_3 \langle \partial_x^2 \theta_{2h}^n, \chi_{2h} \rangle + \gamma_4 \langle \partial_x^2 \vartheta_{2h}^n, \chi_{2h} \rangle = 0, \\ \dfrac{c_1}{\Delta t} \langle \theta_{1h}^n - \theta_{1h}^{n-1}, \chi_{3h} \rangle + \kappa_1 \langle \partial_x \theta_{1h}^n, \partial_x \chi_{3h} \rangle \\ \qquad + \dfrac{d_1}{\Delta t} \langle \vartheta_{1h}^n - \vartheta_{1h}^{n-1}, \chi_{3h} \rangle - \gamma_1 \langle \partial_x v_h^n, \chi_{3h} \rangle = 0, \\ \dfrac{d_1}{\Delta t} \langle \theta_{1h}^n - \theta_{1h}^{n-1}, \chi_{4h} \rangle + \nu_1 \langle \partial_x \vartheta_{1h}^n, \partial_x \chi_{4h} \rangle \\ \qquad + \dfrac{r_1}{\Delta t} \langle \vartheta_{1h}^n - \vartheta_{1h}^{n-1}, \chi_{4h} \rangle - \gamma_2 \langle \partial_x v_h^n, \chi_{4h} \rangle = 0, \\ \dfrac{c_2}{\Delta t} \langle \theta_{2h}^n - \theta_{2h}^{n-1}, \chi_{5h} \rangle + \kappa_2 \langle \partial_x \theta_{2h}^n, \partial_x \chi_{5h} \rangle \\ \qquad + \dfrac{d_2}{\Delta t} \langle \vartheta_{2h}^n - \vartheta_{2h}^{n-1}, \chi_{5h} \rangle + \gamma_3 \langle \partial_x y_h^n, \partial_x \chi_{5h} \rangle = 0, \\ \dfrac{d_2}{\Delta t} \langle \theta_{2h}^n - \theta_{2h}^{n-1}, \chi_{6h} \rangle + \nu_2 \langle \partial_x \vartheta_{2h}^n, \partial_x \chi_{6h} \rangle \\ \qquad + \dfrac{r_2}{\Delta t} \langle \vartheta_{2h}^n - \vartheta_{2h}^{n-1}, \chi_{6h} \rangle + \gamma_4 \langle \partial_x y_h^n, \partial_x \chi_{6h}^n \rangle = 0, \end{cases} \tag{1.23}$$

with $u_h^n = u_h^{n-1} + \Delta t v_h^n$ and $w_h^n = w_h^{n-1} + \Delta t y_h^n$. Here, z_h^0 is an adequate approximation to z^0. Let us introduce the discrete energy given by

$$(\mathcal{E}_h^n)'(t) = \frac12 \Big[\|v_h^n\|^2 + \gamma^2 \|\partial_x^2 w_h^n\|^2 + \|\Phi_\gamma^{\frac12} y_h^n\|^2 + \|\Psi_h^n\|^2 + A_h^n(t) \Big]' = -\kappa_i \|\partial_x \theta_{ih}^n\|^2 - \nu_i \|\partial_x \vartheta_{ih}^n\|^2, \tag{1.24}$$

where

$$A_h^n(t) := c_i \|\theta_{ih}^n\|^2 + 2 d_i \Re \langle \theta_{ih}^n, \vartheta_{ih}^n \rangle + r_i \|\vartheta_{ih}^n\|^2.$$

2 Semigroup approach

Problem (1.11)–(1.13) can be viewed as a Cauchy problem in the Hilbert space $\mathcal{H}$,

$$\begin{cases} \mathcal{A}z' = \mathcal{B}z + \mathcal{F}(z), \\ z(0) = z^0 = (u^0, v^0, \theta_1^0, \vartheta_1^0, w^0, y^0, \theta_2^0, \vartheta_2^0)^T \in \mathcal{H}. \end{cases} \tag{2.1}$$

We define the linear operator $\mathcal{A}$ as follows:

$$\mathcal{A}z = (u, v, \theta_1, \vartheta_1, w, \Phi_\gamma y, \theta_2, \vartheta_2)^T.$$

The linear operator $\mathcal{B}$ and the nonlinear term $\mathcal{F}(z)$ are defined by

$$\mathcal{B}z = \begin{pmatrix} v \\ \partial_x^2 u + \gamma_1 \partial_x \theta_1 + \gamma_2 \partial_x \vartheta_1 \\ -\delta_1^{-1}\big[(d_1\gamma_2 - r_1\gamma_1)\partial_x v - r_1\kappa_1 \partial_x^2 \theta_1 + d_1\nu_1 \partial_x^2 \vartheta_1\big] := \Gamma_1 \\ -\delta_1^{-1}\big[(d_1\gamma_1 - c_1\gamma_2)\partial_x v + d_1\kappa_1 \partial_x^2 \theta_1 - c_1\nu_1 \partial_x^2 \vartheta_1\big] := \Gamma_2 \\ y \\ -\gamma^2 \partial_x^4 w - \gamma_3 \partial_x^2 \theta_2 - \gamma_4 \partial_x^2 \vartheta_2 \\ -\delta_2^{-1}\big[(d_2\gamma_4 - r_2\gamma_3)\partial_x^2 y - r_2\kappa_2 \partial_x^2 \theta_2 + d_2\nu_2 \partial_x^2 \vartheta_2\big] := \Gamma_3 \\ -\delta_2^{-1}\big[(d_2\gamma_3 - c_2\gamma_4)\partial_x^2 y + d_2\kappa_2 \partial_x^2 \theta_2 - c_2\nu_2 \partial_x^2 \vartheta_2\big] := \Gamma_4 \end{pmatrix}, \tag{2.2}$$

$$\mathcal{F}(z) = \begin{pmatrix} 0 \\ \frac{1}{2}\partial_x(\partial_x w)^2 \\ 0 \\ 0 \\ 0 \\ \partial_x(\Psi(u,w)\partial_x w) \\ 0 \\ 0 \end{pmatrix}.$$

The operator $\mathcal{B}$ is associated with the domain

$$\mathcal{D}(\mathcal{A}^{-1}\mathcal{B}) = \left\{ z \in \mathcal{H} : \begin{array}{l} \partial_x^2 u \in L^2(0,L), \quad v \in H_*^1(0,L), \\ w \in H_0^2(0,L) \cap H^4(0,L), \\ y \in H_0^1(0,L), \quad \theta_i, \vartheta_i \in H_0^1(0,L). \end{array} \right\}.$$

3 Main results

Theorem 3.1. *For all $z^0 \in \mathcal{H}$, there exists a unique weak solution $z \in C(\mathbb{R}^+; \mathcal{H})$ of problem (2.1) if and only if the operator $\mathcal{A}^{-1}\mathcal{B}$ is a generator of a semigroup of contractions in $\mathcal{H}$, and the function $\mathcal{A}^{-1}\mathcal{F}(z)$ is locally Lipschitz continuous in $\mathcal{H}$.*

Theorem 3.2. *For $\delta_i, R > 0$ and $z^0 \in \mathsf{B}(0,R)$, the energy (1.19) approaches zero exponentially when time approaches infinity if*

$$\varpi_1 = \gamma_1 - \frac{d_1\gamma_2}{r_1} > 0 \ \text{ and } \ \varpi_2 = \gamma_3 - \frac{d_2\gamma_4}{r_2} > 0,$$

where $\mathsf{B}(0,R)$ is the ball of radius R and δ_i is derived from assumption (1.16).

Theorem 3.3. *Let the assumptions of Theorem 3.2 hold. Then the discrete energy (1.24) approaches zero exponentially when time approaches infinity, i.e.,*

$$\frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t} \le 0, \quad n = 1, \dots, N. \tag{3.1}$$

4 Proof

4.1 Proof of Theorem 3.1

The proof is based on the semigroup method illustrated in the book [11]. First of all, it is clear that $\mathcal{D}(\mathcal{A}^{-1}\mathcal{B})$ is dense in $\mathcal{H}$, i.e.,

$$\overline{\mathcal{D}(\mathcal{A}^{-1}\mathcal{B})} = \mathcal{H}. \tag{4.1}$$

Secondly, $\mathcal{A}^{-1}\mathcal{B}$ is formed by the diagonal matrix of the thermoelasticity with diffusion mass operator $\mathcal{B}_1$ and the thermoplates with mass diffusion operator $\mathcal{B}_2$ which are given by

$$\mathcal{B}_1 = \begin{pmatrix} 0 & I & 0 & 0 \\ \partial_x^2 & 0 & \gamma_1\partial_x & \gamma_2\partial_x \\ 0 & -\delta_1^{-1}(d_1\gamma_2 - r_1\gamma_1)\partial_x & \delta_1^{-1}r_1\kappa_1\partial_x^2 & -\delta_1^{-1}d_1\nu_1\partial_x^2 \\ 0 & -\delta_1^{-1}(d_1\gamma_1 - c_1\gamma_2)\partial_x & -\delta_1^{-1}d_1\kappa_1\partial_x^2 & \delta_1^{-1}c_1\nu_1\partial_x^2 \end{pmatrix},$$

$$\mathcal{B}_2 = \begin{pmatrix} 0 & I & 0 & 0 \\ -\gamma^2\Phi_\gamma^{-1}\partial_x^4 & 0 & -\gamma_3\Phi_\gamma^{-1}\partial_x^2 & -\gamma_4\Phi_\gamma^{-1}\partial_x^2 \\ 0 & -\delta_2^{-1}(d_2\gamma_4 - r_2\gamma_3)\partial_x^2 & \delta_2^{-1}r_2\kappa_2\partial_x^2 & -\delta_2^{-1}d_2\nu_2\partial_x^2 \\ 0 & -\delta_2^{-1}(d_2\gamma_3 - c_2\gamma_4)\partial_x^2 & -\delta_2^{-1}d_2\kappa_2\partial_x^2 & \delta_2^{-1}c_2\nu_2\partial_x^2 \end{pmatrix}.$$

For any $z_1 = (u, v, \theta_1, \vartheta_1), z_2 = (w, y, \theta_2, \vartheta_2) \in \mathcal{D}(\mathcal{A}^{-1}\mathcal{B})$, we have

$$\langle \mathcal{B}_1 z_1, z_1\rangle_{\mathcal{H}} = \langle \partial_x v, \partial_x \overline{u}\rangle + \langle \partial_x^2 u, \overline{v}\rangle + \gamma_1\langle \partial_x\theta_1, \overline{v}\rangle + \gamma_2\langle \partial_x\vartheta_1, \overline{v}\rangle + \left\langle \Lambda_1 \begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}, \begin{pmatrix}\overline{\theta}_1\\ \overline{\vartheta}_1\end{pmatrix}\right\rangle, \tag{4.2}$$

$$\langle z_1, \mathcal{B}_1 z_1\rangle_{\mathcal{H}} = \langle \partial_x u, \partial_x \overline{v}\rangle + \langle v, \partial_x^2 \overline{u}\rangle + \gamma_1\langle v, \partial_x\overline{\theta}_1\rangle + \gamma_2\langle v, \partial_x\overline{\vartheta}_1\rangle + \left\langle \Lambda_1 \begin{pmatrix}\theta_1\\ \vartheta_1\end{pmatrix}, \begin{pmatrix}\overline{\Gamma}_1\\ \overline{\Gamma}_2\end{pmatrix}\right\rangle. \tag{4.3}$$

Adding the previous scalar products (4.2) and (4.3), we get

$$2\Re\langle \mathcal{B}_1 z_1, z_1\rangle_{\mathcal{H}} = -2\kappa_1\|\partial_x\theta_1\|^2 - 2\nu_1\|\partial_x\vartheta_1\|^2 \le 0. \tag{4.4}$$

On the other hand, we have

$$e\langle \mathcal{B}_2 z_2, z_2\rangle_{\mathcal{H}} = \gamma^2\langle \partial_x^2 y, \partial_x^2\overline{w}\rangle - \gamma^2\langle \partial_x^4 w, \overline{y}\rangle - \gamma_3\langle \partial_x^2\theta_2, \overline{y}\rangle - \gamma_4\langle \partial_x^2\vartheta_2, \overline{y}\rangle + \left\langle \Lambda_2 \begin{pmatrix}\Gamma_3\\ \Gamma_4\end{pmatrix}, \begin{pmatrix}\overline{\theta}_2\\ \overline{\vartheta}_2\end{pmatrix}\right\rangle, \tag{4.5}$$

$$\langle z_2, \mathcal{B}_2 z_2\rangle_{\mathcal{H}} = \gamma^2\langle \partial_x^2 w, \partial_x^2\overline{y}\rangle - \gamma^2\langle y, \partial_x^4\overline{w}\rangle - \gamma_3\langle y, \partial_x^2\overline{\theta}_2\rangle - \gamma_4\langle y, \partial_x^2\overline{\vartheta}_2\rangle + \left\langle \Lambda_2 \begin{pmatrix}\theta_2\\ \vartheta_2\end{pmatrix}, \begin{pmatrix}\overline{\Gamma}_3\\ \overline{\Gamma}_4\end{pmatrix}\right\rangle. \tag{4.6}$$

Adding the previous scalar products (4.5) and (4.6), we get

$$2\Re\langle \mathcal{B}_2 z_2, z_2\rangle_{\mathcal{H}} = -2\kappa_2\|\partial_x\theta_2\|^2 - 2\nu_2\|\partial_x\vartheta_2\|^2 \le 0. \tag{4.7}$$

So, from (4.7) and (4.4), we conclude that

$$\langle \mathcal{A}^{-1}\mathcal{B}z, z\rangle = -\kappa_i\|\partial_x\theta_i\|^2 - \nu_i\|\partial_x\vartheta_i\|^2 \le 0. \tag{4.8}$$

Finally, we conclude that the operator from (4.8) and (4.1) is dissipative.

Now, we prove that $I - \mathcal{B}_1$ is onto. For any $\sigma := (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathcal{H}$, we have

$$(I - \mathcal{B}_1)z_1 = \sigma. \tag{4.9}$$

Equation (4.9) gives the following system:

$$\begin{cases} u - v = \sigma_1, \\ v - \partial_x^2 u - \gamma_1\partial_x\theta_1 - \gamma_2\partial_x\vartheta_1 = \sigma_2, \\ \delta_1\theta_1 + (d_1\gamma_2 - r_1\gamma_1)\partial_x v - r_1\kappa_1\partial_x^2\theta_1 + d_1\nu_1\partial_x^2\vartheta_1 = \delta_1\sigma_3, \\ \delta_1\vartheta_1 + (d_1\gamma_1 - c_1\gamma_2)\partial_x v + d_1\kappa_1\partial_x^2\theta_1 - c_1\nu_1\partial_x^2\vartheta_1 = \delta_1\sigma_4, \end{cases} \tag{4.10}$$

Substituting the first equation of system (4.10) we arrive at

$$\begin{cases}(I-\partial_x^2)u-\gamma_1\partial_x\theta_1-\gamma_2\partial_x\vartheta_1=\sigma_2+\sigma_1,\\ (\delta_1-r_1\kappa_1\partial_x^2)\theta_1+(d_1\gamma_2-r_1\gamma_1)\partial_x u+d_1\nu_1\partial_x^2\vartheta_1=\delta_1\sigma_3+(d_1\gamma_2-r_1\gamma_1)\partial_x\sigma_1,\\ (\delta_1-c_1\nu_1\partial_x^2)\vartheta_1+(d_1\gamma_1-c_1\gamma_2)\partial_x u+d_1\kappa_1\partial_x^2\theta_1=\delta_1\sigma_4+(d_1\gamma_1-c_1\gamma_2)\partial_x\sigma_1.\end{cases} \tag{4.11}$$

For $\widetilde{z}_1=(\widetilde{u},\widetilde{\theta}_1,\widetilde{\vartheta}_1)\in\mathcal{H}$, we can extract the bilinear form $\mathfrak{B}_1\in\mathcal{H}\times\mathcal{H}$

$$\begin{aligned}\mathfrak{B}_1(z_1,\widetilde{z}_1)&=\langle u,\overline{\widetilde{u}}\,\rangle+\left\langle\partial_x u+\gamma_1\theta_1+\gamma_2\partial_x\vartheta_1,\partial_x\overline{\widetilde{u}}\,\right\rangle\\ &\quad+\left\langle\delta_1\theta_1+(d_1\gamma_2-r_1\gamma_1)\partial_x u+d_1\nu_1\partial_x^2\vartheta_1,\overline{\widetilde{\theta}}_1\right\rangle+r_1\kappa_1\langle\partial_x\theta_1,\partial_x\overline{\widetilde{\theta}}_1\rangle\\ &\quad+c_1\nu_1\langle\partial_x\vartheta_1,\partial_x\overline{\widetilde{\vartheta}}_1\rangle+\left\langle\delta_1\vartheta_1+(d_1\gamma_1-c_1\gamma_2)\partial_x u+d_1\kappa_1\partial_x^2\theta_1,\overline{\widetilde{\vartheta}}_1\right\rangle,\end{aligned} \tag{4.12}$$

and the linear form $\mathfrak{L}_1\in\mathcal{H}$

$$\mathfrak{L}_1(\widetilde{z}_1)=\langle\sigma_2+\sigma_1,\overline{\widetilde{u}}\,\rangle+\left\langle\delta_1\sigma_3+(d_1\gamma_2-r_1\gamma_1)\partial_x\sigma_1,\overline{\widetilde{\theta}}_1\right\rangle+\left\langle\delta_1\sigma_4+(d_1\gamma_1-c_1\gamma_2)\partial_x\sigma_1,\overline{\widetilde{\vartheta}}_1\right\rangle. \tag{4.13}$$

The bilinear form (4.12) is continuous and coercive and the linear form (4.13) is continuous. Using the Lax–Milgram theorem, we conclude that there exists only one solution satisfying

$$\mathfrak{B}_1(z_1,\widetilde{z}_1)=\mathfrak{L}_1(\widetilde{z}_1),\ \ \forall\,\widetilde{z}_1\in\mathcal{H},$$

such that

$$z_1\in H_*^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L).$$

Now, from (4.10), we have $v\in H_*^1(0,L)$. Then, from (4.11), we have $\partial_x^2 u\in L^2(0,L)$. Thus $z_1\in\mathcal{D}(\mathcal{A}^{-1}\mathcal{B})$ such that $I-\mathcal{B}_1$ is onto.

Similarly to the previous procedure, we shall prove that $I-\mathcal{B}_2$ is onto. For any $\varsigma:=(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4)\in\mathcal{H}$ we have

$$(I-\mathcal{B}_2)z_2=\varsigma,$$

i.e.,

$$\begin{cases}w-y=\varsigma_1,\\ y+\gamma^2\Phi_\gamma^{-1}\partial_x^4 w+\gamma_3\Phi_\gamma^{-1}\partial_x^2\theta_2+\gamma_4\Phi_\gamma^{-1}\partial_x^2\vartheta_2=\varsigma_2,\\ \delta_2\theta_2+(d_2\gamma_4-r_2\gamma_3)\partial_x^2 y-r_2\kappa_2\partial_x^2\theta_2+d_2\nu_2\partial_x^2\vartheta_2=\delta_2\varsigma_3,\\ \delta_2\vartheta_2+(d_2\gamma_3-c_2\gamma_4)\partial_x^2 y+d_2\kappa_2\partial_x^2\theta_2-c_2\nu_2\partial_x^2\vartheta_2=\delta_2\varsigma_4,\end{cases} \tag{4.14}$$

Substitution of the first equation of system (4.14) gives

$$\begin{cases}(I+\gamma^2\Phi_\gamma^{-1}\partial_x^4)w+\gamma_3\Phi_\gamma^{-1}\partial_x^2\theta_2+\gamma_4\Phi_\gamma^{-1}\partial_x^2\vartheta_2=\varsigma_2+\varsigma_1,\\ (\delta_2-r_2\kappa_2\partial_x^2)\theta_2+(d_2\gamma_4-r_2\gamma_3)\partial_x^2 w+d_2\nu_2\partial_x^2\vartheta_2=\delta_2\varsigma_3+(d_2\gamma_4-r_2\gamma_3)\partial_x\varsigma_1,\\ (\delta_2-c_2\nu_2\partial_x^2)\vartheta_2+(d_2\gamma_3-c_2\gamma_4)\partial_x^2 w+d_2\kappa_2\partial_x^2\theta_2=\delta_2\varsigma_4+(d_2\gamma_3-c_2\gamma_4)\partial_x\varsigma_1.\end{cases} \tag{4.15}$$

For $\widetilde{z}_2=(\widetilde{w},\widetilde{\theta}_2,\widetilde{\vartheta}_2)\in\mathcal{H}$, the bilinear form $\mathfrak{B}_2\in\mathcal{H}\times\mathcal{H}$ and the linear form $\mathfrak{L}_2\in\mathcal{H}$ are:

$$\begin{aligned}\mathfrak{B}_2(z_2,\widetilde{z}_2)&=\langle w,\overline{\widetilde{w}}\,\rangle+\left\langle\gamma^2\Phi_\gamma^{-1}\partial_x^4 w+\gamma_3\Phi_\gamma^{-1}\partial_x^2\theta_2+\gamma_4\Phi_\gamma^{-1}\partial_x^2\vartheta_2,\overline{\widetilde{w}}\,\right\rangle\\ &\quad+\left\langle\delta_2\theta_2+(d_2\gamma_4-r_2\gamma_3)\partial_x^2 w+d_2\nu_2\partial_x^2\vartheta_2,\overline{\widetilde{\theta}}_2\right\rangle\\ &\quad+\left\langle\delta_2\vartheta_2+(d_2\gamma_3-c_2\gamma_4)\partial_x^2 w+d_2\kappa_2\partial_x^2\theta_2,\overline{\widetilde{\vartheta}}_2\right\rangle\\ &\quad+c_2\nu_2\langle\partial_x\vartheta_2,\partial_x\overline{\widetilde{\vartheta}}_2\rangle+r_2\kappa_2\langle\partial_x\theta_2,\partial_x\overline{\widetilde{\theta}}_2\rangle,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\mathfrak{L}_2(\widetilde{z}_2)&=\langle\varsigma_2+\varsigma_1,\overline{\widetilde{w}}\,\rangle+\left\langle\delta_2\varsigma_3+(d_2\gamma_4-r_2\gamma_3)\partial_x\varsigma_1,\overline{\widetilde{\theta}}_2\right\rangle\\ &\quad+\left\langle\delta_2\varsigma_4+(d_2\gamma_3-c_2\gamma_4)\partial_x\varsigma_1,\overline{\widetilde{\vartheta}}_2\right\rangle.\end{aligned} \tag{4.17}$$

By using Remark (1.18), we conclude that the bilinear form (4.16) is continuous and coercive and the linear form (4.17) is continuous. Using the Lax–Milgram theorem, we conclude that there exists only one solution satisfying

$$\mathfrak{B}_2(z_2, \widetilde{z}_2) = \mathfrak{L}_2(\widetilde{z}_2), \ \ \forall\, \widetilde{z}_2 \in \mathcal{H},$$

such that

$$z_2 \in H_0^2(0,L) \times H_0^1(0,L) \times H_0^1(0,L).$$

Now, from (4.14) we have $y \in H_0^2(0,L)$ then from (4.15) we have $w \in H^4(0,L)$. Thus $z_2 \in \mathcal{D}(\mathcal{A}^{-1}\mathcal{B})$ such that $I - \mathcal{B}_2$ is onto.

At this stage, we prove that $\mathcal{A}^{-1}\mathcal{F}(z)$ is locally Lipshitz continuous in $\mathcal{H}$. Using the operator (2.2), we get

$$\big\|\mathcal{A}^{-1}(\mathcal{F}(z) - \mathcal{F}(\widetilde{z}))\big\|_{\mathcal{H}} = \big\|(0, f_1, 0, 0, 0, \Phi_\gamma^{-1} f_2, 0, 0)\big\|_{\mathcal{H}} = \|f_1\| + \|\Phi_\gamma^{-\frac{1}{2}} f_2\|,$$

where

$$f_1 = \frac{1}{2}\,\partial_x\big[(\partial_x w)^2 - (\partial_x \widetilde{w})^2\big]$$

and

$$f_2 = \partial_x\big[(\Psi(u,w)\partial_x w) - (\Psi(\widetilde{u},\widetilde{w})\partial_x \widetilde{w})\big].$$

The following estimate is obtained by applying Minkowsky inequality:

$$\begin{aligned} 2\|f_1\| &= \Big\|\partial_x\big[(\partial_x w - \partial_x \widetilde{w})(\partial_x w + \partial_x \widetilde{w})\big]\Big\| \\ &= \Big\|(\partial_x^2 w - \partial_x^2 \widetilde{w})(\partial_x w + \partial_x \widetilde{w}) + (\partial_x w - \partial_x \widetilde{w})(\partial_x^2 w + \partial_x^2 \widetilde{w})\Big\| \\ &\le \big\|(\partial_x^2 w - \partial_x^2 \widetilde{w})(\partial_x w + \partial_x \widetilde{w})\big\| + \big\|(\partial_x w - \partial_x \widetilde{w})(\partial_x^2 w + \partial_x^2 \widetilde{w})\big\|. \end{aligned}$$

Thanks to the Cauchy–Schwarz inequality, we get

$$\begin{aligned} 2\|f_1\| &\le \|\partial_x^2(w - \widetilde{w})\|\,\|\partial_x(w + \widetilde{w})\| + \|\partial_x(w - \widetilde{w})\|\,\|\partial_x^2(w + \widetilde{w})\| \\ &\le \partial_x^2(w - \widetilde{w})\|\big(\|\partial_x w\| + \|\partial_x \widetilde{w}\|\big) + \|\partial_x(w - \widetilde{w})\|\big(\|\partial_x^2 w\| + \|\partial_x^2 \widetilde{w}\|\big). \end{aligned}$$

We use the embeddings $L^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^\infty$ to get

$$\begin{aligned} 2\|f_1\| &\le \|\partial_x^2(w - \widetilde{w})\|\big(|\partial_x w| + |\partial_x \widetilde{w}|\big) + |\partial_x(w - \widetilde{w})|\big(\|\partial_x^2 w\| + \|\partial_x^2 \widetilde{w}\|\big) \\ &\le const.\, \|z - \widetilde{z}\|_{\mathcal{H}}\big(\|z\|_{\mathcal{H}} + \|\widetilde{z}\|_{\mathcal{H}}\big). \end{aligned} \tag{4.18}$$

Using Remark (1.18) we obtain the following estimate:

$$\|\Phi_\gamma^{-\frac{1}{2}} f_2\| = \Big\|\Phi_\gamma^{-\frac{1}{2}}\partial_x\big[(\Psi(u,w)\partial_x w) - (\Psi(\widetilde{u},\widetilde{w})\partial_x \widetilde{w})\big]\Big\| \le const.\,\big\|\Psi(u,w)\partial_x w - \Psi(\widetilde{u},\widetilde{w})\partial_x \widetilde{w}\big\|.$$

By Minkowsky's inequality, we get

$$\begin{aligned} \|\Phi_\gamma^{-\frac{1}{2}} f_2\| &\le const.\,\Big\|\Psi(u,w)\partial_x w - \Psi(\widetilde{u},\widetilde{w})\partial_x \widetilde{w} + \Psi(\widetilde{u},\widetilde{w})\partial_x w - \Psi(\widetilde{u},\widetilde{w})\partial_x w\Big\| \\ &\le const.\,\big\|(\Psi(u,w) - \Psi(\widetilde{u},\widetilde{w}))\partial_x w\big\| + \big\|\Psi(\widetilde{u},\widetilde{w})(\partial_x w - \partial_x \widetilde{w})\big\|. \end{aligned} \tag{4.19}$$

We use the embeddings $L^2 \hookrightarrow L^\infty, H^1 \hookrightarrow L^\infty$ and the estimate (1.21) to get

$$\begin{aligned} \|\Phi_\gamma^{-\frac{1}{2}} f_2\| &\le const.\big\|\Psi(u,w) - \Psi(\widetilde{u},\widetilde{w})\big\|\,|\partial_x w| + \|\Psi(\widetilde{u},\widetilde{w})\|\,|\partial_x w - \partial_x \widetilde{w}| \\ &\le const.\,\mathcal{E}(z - \widetilde{z})\big(|\partial_x w| + \|\Psi(\widetilde{u},\widetilde{w})\|\big) \\ &\le const.\,\|z - \widetilde{z}\|_{\mathcal{H}}\big(\|z\|_{\mathcal{H}} + \|\widetilde{z}\|_{\mathcal{H}}\big). \end{aligned}$$

For $z, \widetilde{z} \in \mathsf{B}(0,R)$, $R > 0$, from (4.18) and (4.19) we have

$$\|\mathcal{A}^{-1}(\mathcal{F}(z) - \mathcal{F}(\widetilde{z}))\|_{\mathcal{H}} \le const.\,\|z - \widetilde{z}\|_{\mathcal{H}}.$$

This the local solution is proven.

For $z^0 \in \mathcal{D}(\mathcal{A}^{-1}\mathcal{B})$, the local solution of (2.1) $z \in \mathcal{D}(\mathcal{A}^{-1}\mathcal{B})$, i.e.,

$$\mathcal{A}^{-1}\mathcal{F}(\mathcal{D}(\mathcal{A}^{-1}\mathcal{B}) \cap \mathsf{B}(0,R)) \subset \mathcal{D}(\mathcal{A}^{-1}\mathcal{B});$$

and satisfies (1.19) or (4.8). Then we have

$$\mathcal{E}(t) \leq \mathcal{E}(0), \quad t \geq 0. \tag{4.20}$$

Using inequalities (1.20) and (4.20), we get

$$\|z\|^2_{\mathcal{H}} \leq 2\mathcal{E}^{\frac{3}{2}}(0), \quad t \geq 0. \tag{4.21}$$

Inequality (4.21) shows the boundedness in the $\mathcal{H}$-norm. So, the global existence is proven.

4.2 Proof of Theorem 3.2

Our argument of proof is based on the following Lyapunov functional:

$$\mathcal{L}(t) := N\mathcal{E}(t) + \sum_{j=1}^{j=3} N_j I_j(t) + \varepsilon I_4(t), \tag{4.22}$$

with $N, N_j > 0$ and ε sufficiently small.

Lemma 4.1. *The functionals I_j, $j = 1, \ldots, 4$, defined by*

$$\begin{aligned}
I_1(t) &:= -\frac{\delta_2}{r_2}\big\langle \theta_2, (-\partial_x^2)^{-1}\Phi_\gamma y\big\rangle,\\
I_2(t) &:= \langle \Phi_\gamma y, w\rangle + \langle u, v\rangle,\\
I_3(t) &:= \frac{\delta_1}{r_1}\langle \theta_1, \phi\rangle,\\
I_4(t) &:= \frac{\delta_2}{r_2}\big\langle \Phi_\gamma y, \varphi\partial_x w\big\rangle,
\end{aligned} \tag{4.23}$$

where

$$\phi = \int_0^x v(k,t)\,dk \quad \text{and} \quad 0 \leq \varphi^2 = \Big(\frac{2x}{L} - 1\Big)^2 \leq 1, \tag{4.24}$$

for all $t \geq 0$ satisfy estimates (4.29), (4.34), (4.38) *and* (4.44).

Under the assumptions of Theorem (3.2), there exists a constant $a > 0$ such that

$$(N - a)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N + a)\mathcal{E}(t) \tag{4.25}$$

with $N > a$ being sufficiently large constant. It follows from the Young, Poincaré, Cauchy–Schwarz inequalities that

$$|\mathcal{L}(t) - N\mathcal{E}(t)| = \Big|\sum_{j=1}^{j=3} I_j(t) + \varepsilon I_4(t)\Big| \leq a\mathcal{E}(t).$$

This proves the validity of the equivalence (4.25).

4.2.1 Functionals estimates

Functional I_1. Taking the derivative of (4.23)$_1$, we get

$$I_1'(t) = -\frac{\delta_2}{r_2}\left\langle \theta_2', (-\partial_x^2)^{-1}\Phi_\gamma y\right\rangle - \frac{\delta_2}{r_2}\left\langle \theta_2, (-\partial_x^2)^{-1}\Phi_\gamma y'\right\rangle = F_1(t) + F_2(t).$$

Using the equations of system (1.11), we obtain

$$\begin{aligned} F_1(t) = &-\Big\langle (-\partial_x^2)^{-1}\Phi_\gamma^{\frac{1}{2}}\left[\kappa_2\partial_x^2\theta_2 - d_2\vartheta_2' + \gamma_3\partial_x^2 y\right], \Phi_\gamma^{\frac{1}{2}} y\Big\rangle \\ &+ \frac{d_2}{r_2}\Big\langle (-\partial_x^2)^{-1}\Phi_\gamma^{\frac{1}{2}}\left[\nu_2\partial_x^2\vartheta_2 - r_2\vartheta_2' + \gamma_4\partial_x^2 y\right], \Phi_\gamma^{\frac{1}{2}} y\Big\rangle \\ = &-\varpi_2\|\Phi_\gamma^{\frac{1}{2}} y\|^2 + d_2\cancel{\big\langle (-\partial_x^2)^{-1}\Phi_\gamma^{\frac{1}{2}}\vartheta_2', \Phi_\gamma^{\frac{1}{2}} y\big\rangle} \\ &+ \left(\frac{d_2\nu_2}{r_2} - \kappa_2\right)\big\langle \Phi_\gamma^{\frac{1}{2}}(\theta_2 + \vartheta_2), \Phi_\gamma^{\frac{1}{2}} y\big\rangle - d_2\cancel{\big\langle (-\partial_x^2)^{-1}\Phi_\gamma^{\frac{1}{2}}\vartheta_2', \Phi_\gamma^{\frac{1}{2}} y\big\rangle}. \end{aligned}$$

For the functional F_2, we have

$$\begin{aligned} F_2(t) &= -\frac{\delta_2}{r_2}\left\langle \theta_2, (-\partial_x^2)^{-1}\partial_x(\Psi(u,w)\partial_x w)\right\rangle + \frac{\delta_2}{r_2}\left\langle \theta_2, \gamma^2\partial_x^4 w + \gamma_3\partial_x^2\theta_2 + \gamma_4\partial_x^2\vartheta_2\right\rangle \\ &= -\frac{\delta_2}{r_2}\left\langle \theta_2, (-\partial_x^2)^{-1}\partial_x(\Psi(u,w)\partial_x w)\right\rangle + \frac{\delta_2\gamma^2}{r_2}\left\langle \theta_2, (-\partial_x^2)^{-1}\partial_x^4 w\right\rangle + \frac{\delta_2\gamma_3}{r_2}\|\theta_2\|^2 + \frac{\delta_2\gamma_4}{r_2}\|\vartheta_2\|^2. \end{aligned}$$

By Young inequality, we get

$$\begin{aligned} &\left(\frac{d_2\nu_2}{r_2} - \kappa_2\right)\Big|\big\langle \Phi_\gamma^{\frac{1}{2}}(\theta_2 + \vartheta_2), \Phi_\gamma^{\frac{1}{2}} y\big\rangle\Big| \\ &\qquad = \text{sing}\left(\frac{d_2\nu_2}{r_2} - \kappa_2\right)\big|\langle \theta_2 + \vartheta_2, \Phi_\gamma y\rangle\big| \le \mu_1\|\Phi_\gamma^{\frac{1}{2}} y\|^2 + c(\mu_1)\big(\|\partial_x\theta_2\|^2 + \|\partial_x\vartheta_2\|^2\big) \end{aligned} \tag{4.26}$$

such that $\mu_1 > 0$, which will be chosen later, and $c(\mu_1) = \frac{c}{\mu_1}$. Using integration by parts, boundary conditions (1.12) and Young's inequality, we obtain

$$\begin{aligned} &-\frac{\delta_2}{r_2}\Big|\big\langle \theta_2, (-\partial_x^2)^{-1}\partial_x(\Psi(u,w)\partial_x w)\big\rangle\Big| \\ &\qquad = -\frac{\delta_2}{r_2}\left[\theta_2(-\partial_x^2)^{-1}(\Psi(u,w)\partial_x w)\right]_{x=0}^{x=L} + \frac{\delta_2}{r_2}\left\langle \partial_x\theta_2, (-\partial_x^2)^{-1}(\Psi(u,w)\partial_x w)\right\rangle \\ &\le \frac{\delta_2}{r_2}\left\|\partial_x(-\partial_x^2)^{-1}\theta_2\right\|\|\Psi\partial_x w\| \le \frac{\delta_2}{r_2}\left\|\partial_x(-\partial_x^2)^{-1}\theta_2\right\|\|\Psi\||\partial_x w| \le \varepsilon\|\Psi\|^2 + c(\varepsilon)\|\partial_x\theta_2\|^2 \end{aligned} \tag{4.27}$$

such that $\varepsilon > 0$ and $c(\varepsilon) = \frac{c}{\varepsilon}$. Using integration by parts, from the boundary conditions (1.12) and Young's inequality, it follows that

$$\begin{aligned} \frac{\delta_2\gamma^2}{r_2}\left|\langle \theta_2, (-\partial_x^2)^{-1}\partial_x^4 w\rangle\right| &= \frac{\gamma^2\delta_2}{r_2}\left[\partial_x^3 w(-\partial_x^2)^{-1}\theta_2\right]_{x=0}^{x=L} - \frac{\gamma^2\delta_2}{r_2}\left\langle \partial_x^3 w, (\partial_x)(-\partial_x^2)^{-1}\theta_2\right\rangle \\ &= -\frac{\gamma^2\delta_2}{r_2}\left[\partial_x^2 w(\partial_x)(-\partial_x^2)^{-1}\theta_2\right]_{x=0}^{x=L} + \frac{\gamma^2\delta_2}{r_2}\langle \partial_x^2 w, \theta_2\rangle \\ &\le \varepsilon\|\partial_x^2 w\|^2 + c(\varepsilon)\|\partial_x\theta_2\|^2 + \frac{\gamma^2\delta_2\varepsilon}{2r_2}\left[(\partial_x^2 w)^2\right]_{x=0}^{x=L} + \pi(t) \end{aligned} \tag{4.28}$$

such that $\varepsilon > 0$ and $c(\varepsilon) = \frac{c}{\varepsilon}$. Note that

$$\pi(t) = \frac{\gamma^2\delta_2}{2r_2\varepsilon}\left[\left((\partial_x)(-\partial_x^2)^{-1}\theta_2\right)^2\right]_{x=0}^{x=L}.$$

From (1.18), we obtain

$$|\pi(t)| \le c(\varepsilon, R)\|\partial_x\theta_2\|^2$$

such that $\varepsilon, R > 0$ and $c(\varepsilon, R) = \frac{c}{R\varepsilon}$. The boundary conditions (1.12) are used to obtain

$$\left[\theta_2(-\partial_x^2)^{-1}(\Psi(u,w)\partial_x w)\right]_{x=0}^{x=L} = \left[\partial_x^3 w(-\partial_x^2)^{-1}\theta_2\right]_{x=0}^{x=L} = 0.$$

By virtue of the above estimates (4.26), (4.27) and (4.28), we get

$$\begin{aligned} I_1'(t) \leq & -(\varpi_2 - \mu_1)\|\Phi_\gamma^{\frac{1}{2}} y\|^2 + \varepsilon\left(\|\partial_x^2 w\|^2 + \|\Psi\|^2\right) \\ & + c(\varepsilon, \mu_1)\left(\|\partial_x\theta_2\|^2 + \|\partial_x\vartheta_2\|^2\right) + c(\varepsilon, R)\|\partial_x\theta_2\|^2. \end{aligned} \tag{4.29}$$

Functional I_2. Taking the derivative of $(4.23)_2$, we obtain

$$I_2'(t) = \langle \Phi_\gamma y', w\rangle + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 + \|v\|^2 + \langle u, v'\rangle = F_3(t) + F_4(t).$$

Using the equations of system (1.11) and the boundary conditions (1.12), we get

$$\begin{aligned} F_3(t) &= \|\Phi_\gamma^{\frac{1}{2}} y\|^2 + \big\langle w, \partial_x(\Psi(u,w)\partial_x w)\big\rangle - \gamma^2\big\langle w, \partial_x^4 w\big\rangle - \gamma_3\langle \partial_x^2\theta_2, w\rangle - \gamma_4\langle\partial_x^2\vartheta_2, w\rangle \\ &= \|\Phi_\gamma^{\frac{1}{2}} y\|^2 - \big\langle(\partial_x w)^2, \Psi(u,w)\big\rangle + \big[w\Psi(u,w)\partial_x w\big]_{x=0}^{x=L} \\ &\quad - \gamma^2\|\partial_x^2 w\|^2 + \gamma_3\langle\partial_x\theta_2, \partial_x w\rangle - \gamma_3\big[\partial_x\theta_2 w\big]_{x=0}^{x=L} + \gamma_4\langle\partial_x\vartheta, \partial_x w\rangle - \gamma_4\big[\partial_x\vartheta_2 w\big]_{x=0}^{x=L}. \end{aligned} \tag{4.30}$$

For the functional F_4, we have

$$F_4(t) = \|v\|^2 - \langle\partial_x u, \Psi(u,w)\rangle + \big[u\Psi(u,w)\big]_{x=0}^{x=L} + \big\langle u, \gamma_1\partial_x\theta_1 + \gamma_2\partial_x\vartheta_1)\big\rangle \tag{4.31}$$

Summing up (4.30) and (4.31), we obtain

$$\begin{aligned} I_2'(t) \leq & -\|\Psi\|^2 - \gamma^2\|\partial_x^2 w\|^2 + \|v\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 \\ & + \big\langle u, \gamma_1\partial_x\theta_1 + \gamma_2\partial_x\vartheta_1)\big\rangle + \big\langle\gamma_3\partial_x\theta_2 + \gamma_4\partial_x\vartheta_2, \partial_x w\big\rangle. \end{aligned} \tag{4.32}$$

By Young inequality, we get

$$\begin{aligned} \big|\big\langle u, \gamma_1\partial_x\theta_1 + \gamma_2\partial_x\vartheta_1\big\rangle\big| &\leq \mu_2\left(\|\Psi\|^2 + \|\partial_x^2 w\|^2\right) + c(\mu_2)\left(\|\partial_x\theta_1\|^2 + \|\partial_x\vartheta_1\|^2\right), \\ \big|\big\langle\gamma_3\partial_x\theta_2 + \gamma_4\partial_x\vartheta_2, \partial_x w\big\rangle\big| &\leq \mu_3\|\partial_x^2 w\|^2 + c(\mu_3)\left(\|\partial_x\theta_2\|^2 + \|\partial_x\vartheta_2\|^2\right) \end{aligned} \tag{4.33}$$

such that μ_2, μ_3 are the positive constants that will be chosen later and $c(\mu_2) = \frac{c}{\mu_2}$, $c(\mu_3) = \frac{c}{\mu_3}$. The boundary conditions (1.12) were used to obtain

$$\big[w\Psi(u,w)\partial_x w\big]_{x=0}^{x=L} = \big[\partial_x\theta_2 w\big]_{x=0}^{x=L} = \big[\partial_x\vartheta_2 w\big]_{x=0}^{x=L} = \big[u\Psi(u,w)\big]_{x=0}^{x=L} = 0.$$

The estimates (4.32) and (4.33) give

$$\begin{aligned} I_2'(t) \leq & -(1-\mu_2)\|\Psi\|^2 - (\gamma^2 - \mu_2 - \mu_3)\|\partial_x^2 w\|^2 + \|v\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2 \\ & + c(\mu_2, \mu_3)\Big(\|\partial_x\theta_2\|^2 + \|\partial_x\vartheta_2\|^2 + \|\partial_x\theta_1\|^2 + \|\partial_x\vartheta_1\|^2\Big). \end{aligned} \tag{4.34}$$

Functional I_3. Taking the derivative of $(4.23)_3$, we get

$$I_3'(t) = \frac{\delta_1}{r_1}\langle\theta_1', \phi\rangle + \frac{\delta_1}{r_1}\langle\theta_1, \phi'\rangle = F_5(t) + F_6(t).$$

From the equations of system (1.11), we arrive at

$$\begin{aligned} F_5(t) &= \left(\kappa_1 - \frac{d_1\nu_1}{r_1}\right)\langle\partial_x^2\theta, \phi\rangle - d_1\langle\vartheta_1', \phi\rangle + \varpi_1\langle\partial_x v, \phi\rangle + d_1\langle\vartheta_1', \phi\rangle \\ &= -\varpi_1\|v\|^2 + \varpi_1[v\phi]_{x=0}^{x=L} + \varpi_1\langle v(0,t), v\rangle + d_1\cancel{\langle\vartheta_1', \phi\rangle} + \left(\kappa_1 - \frac{d_1\nu_1}{r_1}\right)\big[\partial_x\theta_1\phi\big]_{x=0}^{x=L} \\ &\qquad - d_1\cancel{\langle\vartheta_1', \phi\rangle} + \left(\frac{d_1\nu_1}{r_1} - \kappa_1\right)\langle\partial_x\theta_1, \partial_x\phi\rangle. \end{aligned}$$

Using the boundary conditions (1.12) and taking into account that $v \in L^2_*(0,L)$, we have

$$\varpi_1[v\phi]_{x=0}^{x=L} + \varpi_1\langle v(0,t), v\rangle = \big[\partial_x\theta_1\phi\big]_{x=0}^{x=L} = 0.$$

For the functional F_6, we have

$$F_6(t) = \frac{\delta_1}{r_1}\langle\theta_1, \Psi(u,w)\rangle + \frac{\delta_1\gamma_1}{r_1}\|\theta_1\|^2 + \frac{\delta_1\gamma_2}{r_1}\langle\theta_1, \vartheta_1\rangle.$$

Applying Young' inequality we obtain the following estimates:

$$\begin{aligned}\Big(\frac{d_1\nu_1}{r_1} - \kappa_1\Big)\big|\langle\partial_x\theta_1, \partial_x\phi\rangle\big| &= \text{sing}\Big(\frac{d_1\nu_1}{r_1} - \kappa_1\Big)\langle\partial_x\theta_1, v\rangle + \kappa_1 v(0,t)(\theta_1(L) - \theta_1(0)) \\ &\le \mu_4\|v\|^2 + c(\mu_4)\|\partial_x\theta_1\|^2, \qquad (4.35)\end{aligned}$$

also, we have

$$\big|\langle\theta_1, \Psi(u,w)\rangle\big| \le \varepsilon\|\Psi\|^2 + c(\varepsilon)\|\partial_x\theta_1\|^2 \qquad (4.36)$$

and

$$\big|\langle\theta_1, \vartheta_1\rangle\big| \le \varepsilon\|\partial_x\theta_1\|^2 + c(\varepsilon)\|\partial_x\vartheta_1\|^2 \qquad (4.37)$$

such that $\varepsilon > 0$, $\mu_4 > 0$ to be chosen later and $c(\mu_4) = \frac{c}{\mu_4}$. The boundary conditions (1.12) are used to obtain

$$\kappa_1 v(0,t)(\theta_1(L) - \theta_1(0)) = 0.$$

By virtue of inequalities (4.35)–(4.37), I_3' can be expressed as follows:

$$I_3'(t) \le -(\varpi_1 - \mu_4)\|v\|^2 + \varepsilon\|\Psi\|^2 + c(\mu_4, \varepsilon)\big(\|\partial_x\theta_1\|^2 + \|\partial_x\vartheta_1\|^2\big). \qquad (4.38)$$

Functional I_4. Taking the derivative of $(4.23)_4$ and using the equations of system (1.11), we get

$$\begin{aligned}I_4'(t) &= \frac{\delta_2}{r_2}\langle\Phi_\gamma y', \partial_x w\varphi\rangle + \frac{\delta_2}{r_2}\langle\Phi_\gamma y, \partial_x y\varphi\rangle \\ &= \frac{\delta_2}{r_2}\big\langle\partial_x(\Psi(u,w)\partial_x w), \partial_x w\varphi\big\rangle - \frac{\gamma^2\delta_2}{r_2}\langle\partial_x^4 w, \partial_x w\varphi\rangle \\ &\quad - \frac{\gamma_3\delta_2}{r_2}\langle\partial_x^2\theta_2, \partial_x w\varphi\rangle - \frac{\gamma_4\delta_2}{r_2}\langle\vartheta_2, \partial_x w\varphi\rangle + \frac{\delta_2}{r_2}\langle\Phi_\gamma y, \partial_x y\varphi\rangle.\end{aligned}$$

By integration by parts, using the boundary conditions (1.12), (4.24) and (1.18), we obtain

$$\begin{aligned}\frac{\delta_2}{r_2}\big|\big\langle\partial_x(\Psi(u,w)\partial_x w), \partial_x w\varphi\big\rangle\big| &= -\frac{\delta_2}{r_2}\big\langle\Psi(u,w)\partial_x w, \partial_x^2 w\varphi\big\rangle - \frac{2\delta_2}{Lr_2}\big\langle\Psi(u,w), (\partial_x w)^2\big\rangle \\ &\le const.\,\|\Psi\|\,\|\partial_x^2 w\|\,|\partial_x w + (\partial_x w)^2| \\ &\le const.\,\big(\|\Psi\|^2 + \|\partial_x^2 w\|^2\big). \qquad (4.39)\end{aligned}$$

Similarly, we use the integration by parts, the boundary conditions (1.12) and (4.24) to get

$$\frac{\gamma^2\delta_2}{r_2}\big|\langle\partial_x^4 w, \partial_x w\varphi\rangle\big| = \frac{\gamma^2\delta_2}{2r_2}\big[(\partial_x^2 w)^2\varphi\big]_{x=0}^{x=L} - \frac{3\gamma^2\delta_2}{Lr_2}\|\partial_x^2 w\|^2. \qquad (4.40)$$

For the next estimates, we use the integration by parts, the boundary conditions (1.12),(4.24) and Young's inequality to obtain

$$\frac{\gamma_3\delta_2}{r_2}\big|\langle\partial_x^2\theta_2, \partial_x w\varphi\rangle\big| = \frac{\gamma_3\delta_2}{r_2}\langle\partial_x\theta_2, \partial_x^2 w\varphi\rangle + \frac{2\gamma_3\delta_2}{Lr_2}\langle\partial_x\theta_2, \partial_x w\rangle \le \varepsilon\|\partial_x^2 w\|^2 + c(\varepsilon)\|\partial_x\theta_2\|^2 \qquad (4.41)$$

and

$$\frac{\gamma_4\delta_2}{r_2}|\langle\vartheta_2, \partial_x w\varphi\rangle| = \frac{\gamma_4\delta_2}{r_2}\langle\partial_x\vartheta_2, \partial_x^2 w\varphi\rangle + \frac{2\gamma_4\delta_2}{Lr_2}\langle\partial_x\vartheta_2, \partial_x w\rangle \le \varepsilon\|\partial_x^2 w\|^2 + c(\varepsilon)\|\partial_x\vartheta_2\|^2 \qquad (4.42)$$

such that $\varepsilon > 0$ and $c(\varepsilon) = \frac{c}{\varepsilon}$. Using (1.18) and (4.24), we get

$$\frac{\delta_2}{r_2}\left|\langle \Phi_\gamma y, \partial_x y\varphi\rangle\right| \leq const.\,\|\Phi_\gamma^{\frac{1}{2}} y\|^2. \tag{4.43}$$

From (4.39)–(4.42) and (4.43), it follows that

$$I_4'(t) \leq const.\left(\|\Psi\|^2 + \|\partial_x^2 w\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y\|^2\right) + \frac{\gamma^2\delta_2}{2r_2}\left[(\partial_x^2 w)^2\varphi\right]_{x=0}^{x=L} + c(\varepsilon)\left(\|\partial_x\theta_2\|^2 + \|\partial_x\vartheta_2\|^2\right). \tag{4.44}$$

Now, using the derivative of (4.22) and gathering all the above calculations (4.29), (4.34), (4.38) and (4.44), we obtain

$$\begin{aligned}\mathcal{L}'(t) \leq & -p_1\|\Phi_\gamma^{\frac{1}{2}} y\|^2 - p_2\|\Psi\|^2 - p_3\|\partial_x^2 w\|^2 - p_4\|v\|^2 \\ & - p_5\left(\|\partial_x\theta_2\|^2 + \|\partial_x\vartheta_2\|^2\right) - p_6\left(\|\partial_x\theta_1\|^2 + \|\partial_x\vartheta_1\|^2\right) - p_7\|\partial_x\theta_2\|^2,\end{aligned}$$

where

$$\begin{aligned}
p_1 &= \left[N_1(\varpi_2 - \mu_1) - N_2 - \varepsilon\right],\\
p_2 &= \left[N_2(1 - \mu_2) - \varepsilon(N_1 + N_3 + 1)\right],\\
p_3 &= \left[N_2(\gamma^2 - \mu_2 - \mu_3) - \varepsilon(N_1 + 1)\right],\\
p_4 &= \left[N_3(\varpi_1 - \mu_4) - N_2\right],\\
p_5 &= \left[N - N_1 c(\varepsilon, \mu_1) - N_2 c(\mu_2, \mu_3) - c(\varepsilon)\varepsilon\right],\\
p_6 &= \left[N - N_2 c(\mu_2, \mu_3) - N_3 c(\mu_4, \varepsilon)\right],\\
p_7 &= \left[N - N_1 c(\varepsilon, R)\right].
\end{aligned}$$

We set

$$0 < \mu_1 = \frac{\varpi_2}{2}, \quad 0 < \mu_2 = \frac{1}{2}, \quad 0 < \mu_4 = \frac{\varpi_1}{2},$$

and choose N_1 large enough such that

$$N_1\frac{\varpi_2}{2} - N_2 > 0,$$

then we choose N_3 large enough such that

$$N_3\frac{\varpi_1}{2} - N_2 > 0.$$

Next, we choose ε and μ_3 small enough such that

$$N_1\frac{\varpi_2}{2} - N_2 - \varepsilon > 0, \quad N_2\frac{1}{2} - \varepsilon(N_1 + N_3 + 1) > 0$$

and

$$N_2\left(\gamma^2 - \frac{1}{2} - \mu_3) - \varepsilon(N_1 + 1\right) > 0,$$

then we take N such that (4.25) remains valid and

$$\begin{aligned}
N - N_1 c(\varepsilon, \mu_1) - N_2 c(\mu_2, \mu_3) - c(\varepsilon)\varepsilon &> 0,\\
N - N_2 c(\mu_2, \mu_3) - N_3 c(\mu_4, \varepsilon) &> 0,\\
N - N_1 c(\varepsilon, R) &> 0.
\end{aligned}$$

Finally,

$$\mathcal{L}'(t) \leq -c(R)\mathcal{E}(t)$$

such that $c(R)$ is a constant related to the radius R of the ball centered at 0. At this point, the proof is complete.

4.3 Proof of Theorem 3.3

Choosing $\chi_{1h} = v_h^n$, $\chi_{2h} = w_h^n$, $\chi_{3h} = \theta_{1h}^n$, $\chi_{4h} = \vartheta_{1h}^n$, $\chi_{5h} = \theta_{2h}^n$ and $\chi_{6h} = \vartheta_{2h}^n$ in (1.23), we obtain

$$\begin{cases} \frac{1}{2\Delta t}\left(\|v_h^n - v_h^{n-1}\|^2 + \|v_h^n\|^2 - \|v_h^{n-1}\|^2\right) + \left\langle \Psi(u_h^n, w_h^n), \partial_x v_h^n \right\rangle \\ \qquad + \gamma_1 \langle \theta_{1h}^n, \partial_x v_h^n \rangle + \gamma_2 \langle \vartheta_{1h}^n, \partial_x v_h^n \rangle = 0, \\ \frac{1}{2\Delta t}\left(\|\Phi_\gamma^{\frac{1}{2}} y_h^n - \Phi_\gamma^{\frac{1}{2}} y_h^{n-1}\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y_h^n\|^2 - \|\Phi_\gamma^{\frac{1}{2}} y_h^{n-1}\|^2\right) + \left\langle (\Psi(u_h^n, w_h^n)\partial_x w_h^n), \partial_x y_h^n \right\rangle \\ \qquad - \gamma_4 \langle \partial_x \vartheta_{2h}^n, \partial_x y_h^n \rangle + \gamma^2 \langle \partial_x^2 w_h^n, \partial_x^2 y_h^n \rangle - \gamma_3 \langle \partial_x \theta_{2h}^n, \partial_x y_h^n \rangle = 0, \\ \frac{c_1}{2\Delta t}\left(\|\theta_{1h}^n - \theta_{1h}^{n-1}\|^2 + \|\theta_{1h}^n\|^2 - \|\theta_{1h}^{n-1}\|^2\right) + \kappa_1 \|\partial_x \theta_{1h}^n\|^2 \\ \qquad + \frac{d_1}{\Delta t} \langle \vartheta_{1h}^n - \vartheta_{1h}^{n-1}, \theta_{1h}^n \rangle - \gamma_1 \langle \partial_x v_h^n, \theta_{1h}^n \rangle = 0, \\ \frac{d_1}{\Delta t} \langle \theta_{1h}^n - \theta_{1h}^{n-1}, \vartheta_{1h}^n \rangle + \nu_1 \|\partial_x \vartheta_{1h}^n\|^2 - \gamma_2 \langle \partial_x v_h^n, \vartheta_{1h}^n \rangle \\ \qquad + \frac{r_1}{2\Delta t}\left(\|\vartheta_{1h}^n - \vartheta_{1h}^{n-1}\|^2 + \|\vartheta_{1h}^n\|^2 - \|\vartheta_{1h}^{n-1}\|^2\right) = 0, \\ \frac{c_2}{2\Delta t}\left(\|\theta_{2h}^n - \theta_{2h}^{n-1}\|^2 + \|\theta_{2h}^n\|^2 - \|\theta_{2h}^{n-1}\|^2\right) + \kappa_2 \|\partial_x \theta_{2h}^n\|^2 \\ \qquad + \frac{d_2}{\Delta t} \langle \vartheta_{2h}^n - \vartheta_{2h}^{n-1}, \theta_{2h}^n \rangle + \gamma_3 \langle \partial_x y_h^n, \partial_x \theta_{2h}^n \rangle = 0, \\ \frac{d_2}{\Delta t} \langle \theta_{2h}^n - \theta_{2h}^{n-1}, \vartheta_{2h}^n \rangle + \nu_2 \|\partial_x \vartheta_{2h}^n\|^2 + \gamma_4 \langle \partial_x y_h^n, \partial_x \vartheta_{2h}^n \rangle \\ \qquad + \frac{r_2}{2\Delta t}\left(\|\vartheta_{2h}^n - \vartheta_{2h}^{n-1}\|^2 + \|\vartheta_{2h}^n\|^2 - \|\vartheta_{2h}^{n-1}\|^2\right) = 0, \end{cases} \tag{4.45}$$

Note that

$$\begin{aligned} \left\langle \Psi(u_h^n, w_h^n), \partial_x v_h^n \right\rangle + \left\langle (\Psi(u_h^n, w_h^n)\partial_x w_h^n), \partial_x y_h^n \right\rangle &= \frac{1}{\Delta t} \left\langle \Psi(u_h^n, w_h^n), \Psi(u_h^n, w_h^n) - \Psi(u_h^{n-1}, w_h^{n-1}) \right\rangle \\ &\geq \frac{1}{2\Delta t}\left(\|\Psi(u_h^n, w_h^n)\|^2 - \|\Psi(u_h^{n-1}, w_h^{n-1})\|^2\right), \\ \gamma^2 \langle \partial_x^2 w_h^n, \partial_x^2 y_h^n \rangle = \frac{\gamma^2}{\Delta t} \left\langle \partial_x^2 w_h^n, \partial_x^2 w_h^n - \partial_x^2 w_h^{n-1} \right\rangle &\geq \frac{\gamma^2}{2\Delta t}\left(\|\partial_x^2 w_h^n\|^2 - \|\partial_x^2 w_h^{n-1}\|^2\right), \\ \langle \vartheta_{1h}^n - \vartheta_{1h}^{n-1}, \theta_{1h}^n \rangle + \langle \theta_{1h}^n - \theta_{1h}^{n-1}, \vartheta_{1h}^n \rangle &= \langle \vartheta_{1h}^n, \theta_{1h}^n \rangle - \langle \vartheta_{1h}^{n-1}, \theta_{1h}^{n-1} \rangle + \left\langle \theta_{1h}^n - \theta_{1h}^{n-1}, \vartheta_{1h}^n - \vartheta_{1h}^{n-1} \right\rangle \\ \langle \vartheta_{2h}^n - \vartheta_{2h}^{n-1}, \theta_{2h}^n \rangle + \langle \theta_{2h}^n - \theta_{2h}^{n-1}, \vartheta_{2h}^n \rangle &= \langle \vartheta_{2h}^n, \theta_{2h}^n \rangle - \langle \vartheta_{2h}^{n-1}, \theta_{2h}^{n-1} \rangle + \left\langle \theta_{2h}^n - \theta_{2h}^{n-1}, \vartheta_{2h}^n - \vartheta_{2h}^{n-1} \right\rangle. \end{aligned} \tag{4.46}$$

Using (4.45) and (4.46), together with (1.16), we obtain

$$\begin{aligned} &\frac{1}{2\Delta t}\left(\|v_h^n - v_h^{n-1}\|^2 + \|v_h^n\|^2 - \|v_h^{n-1}\|^2\right) \\ &\quad + \frac{1}{2\Delta t}\left(\|\Phi_\gamma^{\frac{1}{2}} y_h^n - \Phi_\gamma^{\frac{1}{2}} y_h^{n-1}\|^2 + \|\Phi_\gamma^{\frac{1}{2}} y_h^n\|^2 - \|\Phi_\gamma^{\frac{1}{2}} y_h^{n-1}\|^2\right) + \frac{c_1}{2\Delta t}\left(\|\theta_{1h}^n\|^2 - \|\theta_{1h}^{n-1}\|^2\right) \\ &\qquad + \kappa_1 \|\partial_x \theta_{1h}^n\|^2 + \nu_1 \|\partial_x \vartheta_{1h}^n\|^2 + \kappa_2 \|\partial_x \theta_{2h}^n\|^2 + \nu_2 \|\partial_x \vartheta_{2h}^n\|^2 \\ &\quad + \frac{c_2}{2\Delta t}\left(\|\theta_{2h}^n\|^2 - \|\theta_{2h}^{n-1}\|^2\right) + \frac{r_1}{2\Delta t}\left(\|\vartheta_{1h}^n\|^2 - \|\vartheta_{1h}^{n-1}\|^2\right) \frac{d_2}{\Delta t}\left(\langle \vartheta_{2h}^n, \theta_{2h}^n \rangle - \langle \vartheta_{2h}^{n-1}, \theta_{2h}^{n-1} \rangle\right) \\ &\qquad + \frac{\gamma^2}{2\Delta t}\left(\|\partial_x^2 w_h^n\|^2 - \|\partial_x^2 w_h^{n-1}\|^2\right) + \frac{r_2}{2\Delta t}\left(\|\vartheta_{2h}^n\|^2 - \|\vartheta_{2h}^{n-1}\|^2\right) \\ &\qquad + \frac{1}{2\Delta t}\left(\|\Psi(u_h^n, w_h^n)\|^2 - \|\Psi(u_h^{n-1}, w_h^{n-1})\|^2\right) + \frac{d_1}{\Delta t}\left(\langle \vartheta_{1h}^n, \theta_{1h}^n \rangle - \langle \vartheta_{1h}^{n-1}, \theta_{1h}^{n-1} \rangle\right) \leq 0. \end{aligned}$$

So, using (1.24), we get (3.1). Thus the proof is completed.

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