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BLOW-UP OF SOLUTIONS FOR A VISCOELASTIC KIRCHHOFF EQUATION WITH A LOGARITHMIC NONLINEARITY, DELAY AND BALAKRISHNAN–TAYLOR DAMPING TERMS

Abstract. A nonlinear viscoelastic Kirchhoff-type equation with a logarithmic nonlinearity, dispersion, delay and Balakrishnan–Taylor damping terms is studied. We prove the blow-up of solutions under a suitable hypothesis.

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რეზიუმე. შესწავლილია არაწრფივი ბლანტი დრეკადი კირხჰოფის ტიპის განტოლება ლოგარითმული არაწრფივობით, დისპერსიით, დაგვიანებით და ბალაკრიშნან-ტეილორის ქრობადი წევრებით. შესაფერისი ჰიპოთეზის შემთხვევაში დამტკიცებულია ამონახსნის აფეთქება.

1 Introduction

In the present work, we consider the following Kirchhoff equation:

$$\begin{cases} |u_t|^p u_{tt} - M(t)\Delta u(t) + \int\limits_0^t h(t-\varrho)\Delta u(\varrho)\,d\varrho - \Delta u_{tt}(t) \\ \qquad\qquad +\beta_1 |u_t(t)|^{m-2}u_t(t) + \beta_2 |u_t(t-\tau)|^{m-2}u_t(t-\tau) = u|u|^{\gamma-2}\ln|u|^k, \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \ \text{ in } \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau) \ \text{ in } \Omega\times(0,\tau), \\ u(x,t)=0 \ \text{ in } \partial\Omega\times(0,\infty) \end{cases} \tag{1.1}$$

where

$$M(t) := \Big(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma\big(\nabla u(t), \nabla u_t(t)\big)_{L^2(\Omega)}\Big),$$

and $\Omega \in \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$; $\gamma \geq 2$, ζ_0, ζ_1, σ, β_1, k are positive constants, β_2 is a real number; $p \geq 0$ for $N=1,2$, and $0 \leq p \leq \frac{4}{N-2}$ for $N \geq 3$, and $m \geq 2$ for $N=1,2$, and $2 < m \leq \frac{N+2}{N-2}$ for $N \geq 3$, h is a positive function.

Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory is called viscoelastic damping term, where the kernel of the term of memory is the function h. See [9, 14–16, 19, 20, 22, 23, 30].

In [3], Balakrishnan and Taylor proposed a new model of damping and called it the Balakrishnan-Taylor damping, as it relates to the span problem and the plate equation. For more depth, there are some papers that focused on the study of this damping [2, 3, 7, 11, 15, 21, 23, 29, 31].

The effect of the delay often appears in many applications and practical problems and turns a lot of systems into different problems worth studying. Recently, the stability and the asymptotic behavior of evolution systems with time delay has been studed by many authors (see [10, 14–19, 22, 23, 32]).

The great importance of the logarithmic nonlinearity in physics is that they appear in several issues and theories, including symmetry, cosmology, quantum mechanics, as well as nuclear physics. It is also used in many applications such as optical, nuclear and even subterranean physics. Many researchers also touched on this type of problems in different issues, where the global existence of solutions, stability and blow-up of solutions were studied. For more information, the reader is referred to [5, 6, 8, 11, 13, 15, 24, 25, 27].

Based on all of the above, we believe that the combination of these terms of damping (memory term, Balakrishnan–Taylor damping, logarithmic nonlinearity, dispersion and the delay terms) in one particular problem with the addition of the delay term $(\beta_2|u_t(t-\tau)|^{m-2}u_t(t-\tau))$ constitutes a new problem worthy of study and research, different from the above that we will try to shed light on.

Our paper is divided into several sections. In Section 2, we lay down the hypotheses, concepts and lemmas we need. In Section 3, we state and prove the blow-up of solutions.

2 Preliminaries

To study our problem, in this section, we will need some materials.

First, we introduce the following hypotheses for β_2 and h:

(A1) $h : \mathbb{R}_+ \to \mathbb{R}_+$ are non-increasing C^1 functions satisfying

$$h(t) > 0, \quad \zeta_0 - \int\limits_0^\infty h(\varrho)\,d\varrho = l > 0. \tag{2.1}$$

(A2)

$$|\beta_2| < \beta_1. \tag{2.2}$$

Let us introduce

$$(h \circ \psi)(t) := \int_\Omega \int_0^t h(t-\varrho)|\psi(t)-\psi(\varrho)|^2 \, d\varrho \, dx.$$

As in [32], taking the new variables

$$y(x,\rho,t) = u_t(x, t-\tau\rho), \quad (x,\rho,t) \in \Omega \times (0,1) \times \mathbb{R}_+,$$

which satisfy

$$\begin{cases} \tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0, \\ y(x,0,t) = u_t(x,t), \end{cases} \tag{2.3}$$

problem (1.1) can be written as

$$\begin{cases} |u_t|^p u_{tt} - M(t)\Delta u(t) + \int_0^t h(t-\varrho)\Delta u(\varrho)\, d\varrho - \Delta u_{tt}(t) \\ \qquad\qquad + \beta_1 |u_t(t)|^{m-2} u_t(t) + \beta_2 |y(x,1,t)|^{m-2} y(x,1,t) = u|u|^{\gamma-2} \ln|u|^k, \\ \tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \ \text{ in } \Omega, \\ y(x,\rho,0) = f_0(x,-\tau\rho) \ \text{ in } \Omega \times (0,1), \\ u(x,t) = 0 \ \text{ in } \partial\Omega \times (0,\infty), \end{cases} \tag{2.4}$$

where $(x,\rho,t) \in \Omega \times (0,1) \times (0,\infty)$. Now, we give the energy functional.

Lemma 2.1. *The energy functional E, defined by*

$$E(t) = \frac{1}{p+2}\|u_t\|_{p+2}^{p+2} + \frac{1}{2}\left(\zeta_0 - \int_0^t h(\varrho)\,d\varrho\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2 + \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4$$
$$+ \frac{1}{2}(h \circ \nabla u)(t) + \frac{k}{\gamma}\|u(t)\|_\gamma^\gamma - \frac{1}{\gamma}\int_\Omega |u|^\gamma \ln|u|^k \, dx + \frac{\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m \, d\rho, \tag{2.5}$$

satisfies

$$E'(t) \le -C_0\Big(\|u_t(t)\|_m^m + \|y(x,1,t)\|_m^m\Big)$$
$$+ \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2}h(t)\|\nabla u(t)\|_2^2 - \frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right)^2 \le 0, \tag{2.6}$$

where $\xi > 0$ satisfies

$$\tau(m-1)|\beta_2| \le \xi \le \tau\big(m\beta_1 - |\beta_2|\big).$$

Proof. Taking the inner product of $(2.4)_1$ with u_t and integrating over Ω, we find

$$\big(|u_t|^p u_{tt}(t), u_t(t)\big)_{L^2(\Omega)} - \big(M(t)\Delta u(t), u_t(t)\big)_{L^2(\Omega)}$$
$$- \big(\Delta u_{tt}(t), u_t(t)\big)_{L^2(\Omega)} + \left(\int_0^t h(t-\varrho)\Delta u(\varrho)\,d\varrho, u_t(t)\right)_{L^2(\Omega)} + \beta_1\big(|u_t|^{m-2}u_t, u_t\big)_{L^2(\Omega)}$$
$$+ \beta_2\Big(|y(x,1,t)|^{m-2}y(x,1,t), u_t(t)\Big)_{L^2(\Omega)} - \Big(ku|u|^{\gamma-2}\ln|u|, u_t(t)\Big)_{L^2(\Omega)} = 0. \tag{2.7}$$

A direct calculation gives

$$\big(|u_t|^p u_{tt}(t), u_t(t)\big)_{L^2(\Omega)} = \frac{1}{p+2}\,\frac{d}{dt}\left(\|u_t(t)\|_{p+2}^{p+2}\right), \tag{2.8}$$

$$-\big(\Delta u_{tt}(t), u_t(t)\big)_{L^2(\Omega)} = \frac{1}{2}\,\frac{d}{dt}\left(\|\nabla u_t(t)\|_2^2\right). \tag{2.9}$$

By integration by parts, we find

$$\begin{aligned}
&-\big(M(t)\,\Delta u(t), u_t(t)\big)_{L^2(\Omega)} \\
&\quad= -\Big(\big(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma(\nabla u(t), \nabla u_t(t))_{L^2(\Omega)}\big)\Delta u(t), u_t(t)\Big)_{L^2(\Omega)} \\
&\quad= \Big(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma\big(\nabla u(t), \nabla u_t(t)\big)_{L^2(\Omega)}\Big)\int_\Omega \nabla u(t)\cdot\nabla u_t(t)\,dx \\
&\quad= \Big(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma\big(\nabla u(t), \nabla u_t(t)\big)_{L^2(\Omega)}\Big)\frac{d}{dt}\left\{\int_\Omega |\nabla u(t)|^2\,dx\right\} \\
&\quad= \frac{d}{dt}\left\{\frac{1}{2}\Big(\zeta_0 + \frac{\zeta_1}{2}\,\|\nabla u\|_2^2\Big)\|\nabla u(t)\|_2^2\right\} + \frac{\sigma}{4}\,\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}^2.
\end{aligned} \tag{2.10}$$

So, we have

$$\begin{aligned}
&\bigg(\int_0^t h(t-\varrho)\Delta u(\varrho)\,d\varrho, u_t(t)\bigg)_{L^2(\Omega)} \\
&\qquad= \int_0^t h(t-\varrho)\big(\Delta u(\varrho), u_t(t)\big)_{L^2(\Omega)}\,d\varrho = -\int_0^t h(t-\varrho)\bigg[\int_\Omega \nabla u(x,\varrho)\nabla u(x,t)\,dx\bigg]\,d\varrho
\end{aligned}$$

and

$$-\nabla u(x,\varrho)\cdot\nabla u(x,t) = \frac{1}{2}\,\frac{d}{dt}\left\{\big|\nabla u(x,\varrho) - \nabla u(x,t)(t)\big|^2\right\} - \frac{1}{2}\,\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\}.$$

Then

$$\begin{aligned}
&-\int_0^t h(t-\varrho)\big(\nabla u(\varrho), \nabla u_t(t)\big)_{L^2(\Omega)}\,d\varrho \\
&\quad= -\int_0^t h(t-\varrho)\int_\Omega\left[\frac{1}{2}\,\frac{d}{dt}\left\{\big|\nabla u(x,\varrho) - \nabla u(x,t)\big|^2\right\}\right]dx\,ds \\
&\qquad\qquad - \int_0^t h(t-\varrho)\int_\Omega\left[\frac{1}{2}\,\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\}\right]dx\,d\varrho \\
&\quad= \frac{1}{2}\int_0^t h(t-\varrho)\bigg[\frac{d}{dt}\bigg\{\int_\Omega \big|\nabla u(x,t) - \nabla u(x,\varrho)\big|^2\,dx\bigg\}\bigg]\,d\varrho \\
&\qquad\qquad - \frac{1}{2}\int_0^t h(t-\varrho)\bigg[\frac{d}{dt}\left\{\|\nabla u(x,t)\|_2^2\right\}\bigg]\,dx\,d\varrho.
\end{aligned} \tag{2.11}$$

We use (2.1) to obtain

$$\frac{1}{2}\int_0^t h(t-\varrho)\bigg[\frac{d}{dt}\bigg\{\int_\Omega \big|\nabla u(x,t) - \nabla u(x,\varrho)\big|^2\,dx\bigg\}\bigg]\,d\varrho$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{d}{dt}\left\{ \int_0^t h(t-\varrho)\left[\int_\Omega \left|\nabla u(x,t)-\nabla u(x,\varrho)\right|^2 dx \right] \right\} d\varrho \\
&\qquad - \frac{1}{2}\int_0^t h'(t-\varrho)\left[\int_\Omega \left|\nabla u(x,t)-\nabla u(x,\varrho)\right|^2 dx \right] d\varrho \\
&= \frac{1}{2}\frac{d}{dt}(h\circ\nabla u)(t) - \frac{1}{2}(h'\circ\nabla u)(t) \qquad (2.12)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right] dx\, d\varrho \\
&\qquad = -\frac{1}{2}\left(\int_0^t h(t-\varrho)\, d\varrho\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right) dx \\
&\qquad = -\frac{1}{2}\left(\int_0^t h(\varrho)\, d\varrho\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|_2^2\right\}\right) dx \\
&\qquad = -\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(\varrho)\, d\varrho\right)\|\nabla u(t)\|_2^2\right\} + \frac{1}{2}h(t)\|\nabla u(t)\|_2^2 . \qquad (2.13)
\end{aligned}
$$

By substituting (2.12) and (2.13) into (2.11), we get

$$
\begin{aligned}
&\left(\int_0^t h(t-\varrho)\Delta u(\varrho)\, d\varrho, u_t(t)\right)_{L^2(\Omega)} \\
&\quad = \frac{d}{dt}\left\{\frac{1}{2}(h\circ\nabla u)(t) - \frac{1}{2}\left(\int_0^t h(\varrho)\, d\varrho\right)\|\nabla u(t)\|_2^2\right\} - \frac{1}{2}(h'\circ\nabla u)(t) + \frac{1}{2}h(t)\|\nabla u(t)\|_2^2. \qquad (2.14)
\end{aligned}
$$

Thus we have

$$
-\left(ku|u|^{\gamma-2}\ln|u|, u_t(t)\right)_{L^2(\Omega)} = \frac{d}{dt}\left\{\frac{k}{\gamma}\|u(t)\|_\gamma^\gamma - \frac{1}{\gamma}\int_\Omega |u|^\gamma \ln|u|^k\, dx\right\}. \qquad (2.15)
$$

Now, multiplying equation $(2.4)_2$ by $-y\xi$, integrating over $\Omega\times(0,1)$, and using $(2.3)_2$, we get

$$
\begin{aligned}
&\frac{d}{dt}\frac{\xi}{m}\int_\Omega\int_0^1 |y(x,\rho,t)|^m\, d\rho\, dx = -\left(\frac{\xi}{\tau}\right)\int_\Omega\int_0^1 |y|^{m-1}y_\rho\, d\rho\, dx \\
&= -\frac{\xi}{m\tau}\int_\Omega\int_0^1 \frac{d}{d\rho}|y(x,\rho,s,t)|^m\, d\rho\, dx = \frac{\xi}{m\tau}\int_\Omega \left(|y(x,0,t)|^m - |y(x,1,t)|^m\right) dx \\
&\qquad = \frac{\xi}{m\tau}\left(\int_\Omega |u_t(t)|^m\, dx - \int_\Omega |y(x,1,t)|^m\, dx\right) = \frac{\xi}{m\tau}\left(\|u_t(t)\|_m^m - \|y(x,1,t)\|_m^m\right), \qquad (2.16)
\end{aligned}
$$

and by Young's inequality, we have

$$
\beta_2\left(|y(x,1,t)|^{m-2}y(x,1,t), u_t(t)\right)_{L^2(\Omega)} \le \frac{|\beta_2|}{m}\|u_t(t)\|_m^m + \frac{(m-1)|\beta_2|}{m}\|y(x,1,t)\|_m^m. \qquad (2.17)
$$

Substituting (2.8)–(2.10) and (2.14)–(2.17) into (2.7), we find (2.5) and (2.6), where

$$C_0 = \min\left\{\beta_1 - \frac{\xi}{m\tau} - \frac{|\beta_2|}{m}, \frac{\xi}{m\tau} - \frac{(m-1)|\beta_2|}{m}\right\}. \qquad \square$$

The local existence result for problem (2.4) is stated without providing the proof. Indeed, using the Faedo–Galerkin method and a combination of the works [22, 29, 33], one can prove the theorem below.

Theorem 2.1. *Suppose that (2.1), (2.2) are satisfied. Then for any $u_0, u_1 \in H_0^1(\Omega) \cap L^2(\Omega)$ and $f_0 \in L^2(\Omega, (0,1))$, there exists a weak solution u of problem (2.4) such that*

$$u \in C\big(\,]0,T[\,, H_0^1(\Omega)\big) \cap C^1\big(\,]0,T[\,, L^2(\Omega)\big),$$
$$u_t \in C\big(\,]0,T[\,, H_0^1(\Omega)\big) \cap L^2\big(\,]0,T[\,, L^2(\Omega,(0,1))\big).$$

Lemma 2.2 ([27])**.** *There exists a positive constant $c(\Omega) > 0$ such that*

$$\left(\int_\Omega |u|^\gamma \ln|u|^k\,dx\right)^{\frac{s}{\gamma}} \le c\left(\int_\Omega |u|^\gamma \ln|u|^k\,dx + \|\nabla u\|_2^2\right)$$

for any $2 \le s \le \gamma$, provided that $\int_\Omega |u|^\gamma \ln|u|^k\,dx \ge 0$.

Corollary 2.1 ([27])**.** *There exists a positive constant $c(\Omega) > 0$ such that*

$$\|u\|_2^2 \le c\left[\left(\int_\Omega |u|^\gamma \ln|u|^k\,dx\right)^{\frac{2}{\gamma}} + \|\nabla u\|_2^{\frac{4}{\gamma}}\right],$$

provided that $\int_\Omega |u|^\gamma \ln|u|^k\,dx \ge 0$.

Lemma 2.3 ([27])**.** *There exists a positive constant $c(\Omega) > 0$ such that*

$$\|u\|_\gamma^s \le c\Big(\|u\|_\gamma^\gamma + \|\nabla u\|_2^2\Big)$$

for any $u \in L^\gamma(\Omega)$ and $2 \le s \le \gamma$.

3 Blow-up result

In this section, we prove the blow-up result of the solution of problem (2.4).

First, we define the functional

$$\begin{aligned}\mathbb{H}(t) = -E(t) = &-\frac{1}{p+2}\|u_t\|_{p+2}^{p+2}\\ &-\frac{1}{2}\left(\zeta_0 - \int_0^t h(\varrho)\,d\varrho\right)\|\nabla u(t)\|_2^2 - \frac{1}{2}\|\nabla u_t(t)\|_2^2 - \frac{\zeta_1}{4}\|\nabla u(t)\|_2^4\\ &-\frac{1}{2}(h \circ \nabla u)(t) - \frac{k}{\gamma}\|u(t)\|_\gamma^\gamma + \frac{1}{\gamma}\int_\Omega |u|^\gamma \ln|u|^k\,dx - \frac{\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m\,d\rho. \qquad (3.1)\end{aligned}$$

Theorem 3.1. *Assume (2.1), (2.2) hold and suppose that $E(0) < 0$. Then the solution of problem (2.4) blow-up in finite time.*

Proof. From (2.6) we have

$$E(t) \leq E(0) \leq 0.$$

Therefore,

$$\mathbb{H}'(t) = -E'(t) \geq C_0\Big(\|u_t(t)\|_m^m + \|y(x,1,t)\|_m^m\Big),$$

hence

$$\begin{aligned} \mathbb{H}'(t) &\geq C_0\|u_t(t)\|_m^m \geq 0, \\ \mathbb{H}'(t) &\geq C_0\|y(x,1,t)\|_m^m \geq 0. \end{aligned} \tag{3.2}$$

By (3.1), we have

$$0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) \leq \frac{1}{\gamma}\int_\Omega |u|^\gamma \ln|u|^k\,dx. \tag{3.3}$$

We set

$$\mathcal{K}(t) = \mathbb{H}^{1-\alpha} + \frac{\varepsilon}{p+1}\int_\Omega u|u_t|^p u_t\,dx + \varepsilon\int_\Omega \nabla u\nabla u_t\,dx + \frac{\sigma}{4}\|\nabla u\|_2^4, \tag{3.4}$$

where $\varepsilon > 0$ will be assigned later and

$$\frac{2(\gamma-1)}{\gamma^2} < \alpha < \frac{\gamma-2}{2\gamma} < 1. \tag{3.5}$$

By multiplying $(2.4)_1$ by u and with a derivative of (3.4), we get

$$\begin{aligned} \mathcal{K}'(t) = (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) &+ \frac{\varepsilon}{p+1}\|u_t\|_{p+2}^{p+2} + \varepsilon\|\nabla u_t\|_2^2 + \varepsilon\int_\Omega |u|^\gamma\ln|u|^k\,dx \\ &- \varepsilon\zeta_0\|\nabla u\|_2^2 - \varepsilon\zeta_1\|\nabla u\|_2^4 + \underbrace{\varepsilon\int_\Omega \nabla u\int_0^t h(t-\varrho)\nabla u(\varrho)\,d\varrho\,dx}_{J_1} \\ &\underbrace{- \varepsilon\beta_1\int_\Omega u.u_t.|u_t|^{m-2}\,dx}_{J_2}\ \underbrace{- \varepsilon\beta_2\int_\Omega u\cdot y(x,1,t)\cdot|y(x,1,t)|^{m-2}\,dx}_{J_3}. \end{aligned} \tag{3.6}$$

We have

$$\begin{aligned} J_1 &= \varepsilon\int_0^t h(t-\varrho)\,d\varrho\int_\Omega \nabla u\cdot(\nabla u(\varrho) - \nabla u(t))\,dx\,d\varrho + \varepsilon\int_0^t h(\varrho)\,d\varrho\|\nabla u\|_2^2 \\ &\geq \frac{\varepsilon}{2}\Big(\int_0^t h(\varrho)\,d\varrho\Big)\|\nabla u\|_2^2 - \frac{\varepsilon}{2}(h\circ\nabla u), \end{aligned}$$

and for $\delta_1, \delta_2 > 0$,

$$\begin{aligned} J_2 &\geq -\varepsilon\delta_1\|u\|_2^2 - \varepsilon\frac{c_1}{4\delta_1}\|u\|_m^m, \\ J_3 &\geq -\varepsilon\delta_2\|u\|_2^2 - \varepsilon\frac{c_2}{4\delta_2}\|y(x,1,t)\|_m^m. \end{aligned}$$

From (3.6) we find

$$\begin{aligned} \mathcal{K}'(t) \geq{}& (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{p+1}\|u_t\|_{p+2}^{p+2} + \varepsilon\|\nabla u_t\|_2^2 \\ &+ \varepsilon\int_\Omega |u|^\gamma\ln|u|^k\,dx - \varepsilon\zeta_1\|\nabla u\|_2^4 - \varepsilon\Big[\Big(\zeta_0 - \frac{1}{2}\int_0^t h(\varrho)\,d\varrho\Big)\Big]\|\nabla u\|_2^2 \\ &- \frac{\varepsilon}{2}(h\circ\nabla u) - \varepsilon(\delta_1+\delta_2)\|u\|_2^2 - \varepsilon\frac{c_1}{4\delta_1}\|u\|_m^m - \varepsilon\frac{c_2}{4\delta_2}\|y(x,1,t)\|_m^m. \end{aligned} \tag{3.7}$$

At this point, setting δ_1, δ_1 so that for large κ, which will be be specified later,

$$\frac{c_1}{4C_0\delta_1} = \frac{\kappa\mathbb{H}^{-\alpha}(t)}{2}, \quad \frac{c_2}{4C_0\delta_2} = \frac{\kappa\mathbb{H}^{-\alpha}(t)}{2},$$

due to (3.2) from (3.7) we get

$$\begin{aligned}\mathcal{K}'(t) \geq \big[(1-\alpha) - \varepsilon\kappa\big]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{p+1}\|u_t\|_{p+2}^{p+2} + \varepsilon\|\nabla u_t\|_2^2 - \frac{\varepsilon}{2}(h\circ\nabla u) - \varepsilon\zeta_1\|\nabla u\|_2^4 \\ - \varepsilon\Big(\zeta_0 - \frac{1}{2}\int_0^t h(\varrho)\,d\varrho\Big)\|\nabla u\|_2^2 - \varepsilon\Big(\frac{c_3\mathbb{H}^\alpha(t)}{2C_0\kappa}\Big)\|u\|_2^2 + \varepsilon\int_\Omega |u|^\gamma \ln|u|^k\,dx, \quad (3.8)\end{aligned}$$

where $c_3 = c_1 + c_2$.

Now, for $0 < a < 1$, from (3.1),

$$\begin{aligned}\varepsilon\int_\Omega |u|^\gamma \ln|u|^k\,dx = \varepsilon a\int_\Omega |u|^\gamma \ln|u|^k\,dx + \frac{\varepsilon\gamma(1-a)}{p+2}\|u_t\|_{p+2}^{p+2} + \varepsilon\gamma(1-a)\mathbb{H}(t) \\ + \varepsilon\frac{\gamma(1-a)}{2}\Big(\zeta_0 - \int_0^t h(\varrho)\,d\varrho\Big)\|\nabla u\|_2^2 + \varepsilon\frac{\gamma(1-a)}{2}\|\nabla u_t\|_2^2 + \varepsilon\frac{\zeta_1\gamma(1-a)}{2}\|\nabla u\|_2^4 \\ - \varepsilon\frac{\gamma(1-a)}{2}(h\circ\nabla u) + \varepsilon k(1-a)\|u\|_\gamma^\gamma + \frac{\varepsilon\gamma(1-a)\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m\,d\rho.\end{aligned}$$

Substituting in (3.8), we get

$$\begin{aligned}\mathcal{K}'(t) \geq \big\{(1-\alpha) - \varepsilon\kappa\big\}\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon a\int_\Omega |u|^\gamma \ln|u|^k\,dx \\ + \varepsilon\Big\{\frac{\gamma(1-a)}{p+2} + \frac{1}{p+1}\Big\}\|u_t\|_{p+2}^{p+2} + \varepsilon\Big\{1 + \frac{\gamma(1-a)}{2}\Big\}\|\nabla u_t\|_2^2 \\ + \varepsilon\Big\{\frac{\gamma(1-a)}{2}\Big(\zeta_0 - \int_0^t h(\varrho)\,d\varrho\Big) - \Big(\zeta_0 - \frac{1}{2}\int_0^t h(\varrho)\,d\varrho\Big)\Big\}\|\nabla u\|_2^2 \\ + \varepsilon\zeta_1\Big\{\frac{\gamma(1-a)}{2} - 1\Big\}\|\nabla u\|_2^4 + \varepsilon\Big\{\frac{\gamma(1-a)}{2} - \frac{1}{2}\Big\}(h\circ\nabla u) - \varepsilon\Big(\frac{c_3\mathbb{H}^\alpha(t)}{2C_0\kappa}\Big)\|u\|_2^2 \\ + \varepsilon k(1-a)\|u\|_\gamma^\gamma + \varepsilon\gamma(1-a)\mathbb{H}(t) + \frac{\varepsilon\gamma(1-a)\xi}{m}\int_0^1 \|y(x,\rho,t)\|_m^m\,d\rho. \quad (3.9)\end{aligned}$$

According to (3.3), Corollary 2.1 and Young's inequality, we get

$$\begin{aligned}\mathbb{H}^\alpha(t)\|u\|_2^2 &\leq \Big(\int_\Omega |u|^\gamma \ln|u|^k\,dx\Big)^\alpha \|u\|_2^2 \\ &\leq c\Big[\Big(\int_\Omega |u|^\gamma \ln|u|^k\,dx\Big)^{\alpha+\frac{2}{\gamma}} + \Big(\int_\Omega |u|^\gamma \ln|u|^k\,dx\Big)^\alpha \|\nabla u\|_2^{\frac{4}{\gamma}}\Big] \\ &\leq c\Big[\Big(\int_\Omega |u|^\gamma \ln|u|^k\,dx\Big)^{\frac{(\alpha\gamma+2)}{\gamma}} + \Big(\int_\Omega |u|^\gamma \ln|u|^k\,dx\Big)^{\frac{\alpha\gamma}{(\gamma-2)}} + \|\nabla u\|_2^2\Big]\end{aligned}$$

By (3.5), we have

$$2 < \alpha\gamma + 2 \leq \gamma \ \text{ and } \ 2 < \frac{\alpha\gamma^2}{\gamma-2} \leq \gamma.$$

Hence Lemma 2.2 gives

$$\mathbb{H}^{\alpha}(t)\|u\|_2^2 \leq c\Big(\int\limits_{\Omega} |u|^{\gamma} \ln |u|^{k}\, dx + \|\nabla u\|_2^2\Big). \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\begin{aligned}
\mathcal{K}'(t) \geq{}& \{(1-\alpha) - \varepsilon\kappa\}\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \varepsilon\Big(a - \frac{c_4}{2C_0\kappa}\Big)\int\limits_{\Omega} |u|^{\gamma}\ln|u|^{k}\, dx\\
&+ \varepsilon\Big\{\frac{\gamma(1-a)}{p+2} + \frac{1}{p+1}\Big\}\|u_t\|_{p+2}^{p+2} + \varepsilon\Big\{1 + \frac{\gamma(1-a)}{2}\Big\}\|\nabla u_t\|_2^2\\
&+ \varepsilon\Big\{\frac{\gamma(1-a)}{2}\Big(\zeta_0 - \int\limits_0^t h(\varrho)\, d\varrho\Big) - \Big(\zeta_0 - \frac{1}{2}\int\limits_0^t h(\varrho)\, d\varrho\Big) - \frac{c_4}{2C_0\kappa}\Big\}\|\nabla u\|_2^2\\
&+ \varepsilon\zeta_1\Big\{\frac{\gamma(1-a)}{2} - 1\Big\}\|\nabla u\|_2^4 + \varepsilon\Big\{\frac{\gamma(1-a)}{2} - \frac{1}{2}\Big\}(h \circ \nabla u)\\
&+ \varepsilon k(1-a)\|u\|_{\gamma}^{\gamma} + \varepsilon\gamma(1-a)\mathbb{H}(t) + \frac{\varepsilon\gamma(1-a)\xi}{m}\int\limits_0^1 \|y(x,\rho,t)\|_m^m\, d\rho.
\end{aligned}$$

At this stage, we take $a > 0$ small enough so that

$$\lambda_1 = \frac{\gamma(1-a)}{2} - 1 > 0,$$

and assume

$$\int\limits_0^{\infty} h(\varrho)\, d\varrho < \frac{\frac{\gamma(1-a)}{2} - 1}{\frac{\gamma(1-a)}{2} - \frac{1}{2}} = \frac{2\lambda_1}{2\lambda_1 + 1},$$

which gives

$$\lambda_2 = \Big\{\Big(\frac{\gamma(1-a)}{2} - 1\Big) - \Big(\int\limits_0^t h(\varrho)\, d\varrho\Big)\Big(\frac{\gamma(1-a)}{2} - \frac{1}{2}\Big)\Big\} > 0,$$

then we choose κ so large that

$$\lambda_3 = a - \frac{c_4}{2C_0\kappa} > 0,$$
$$\lambda_4 = \lambda_2 - \frac{c_4}{2C_0\kappa} > 0.$$

Finally, we fix κ, a and appoint ε small enough so that

$$\lambda_5 = (1-\alpha) - \varepsilon\kappa > 0$$

and

$$\mathcal{K}(0) > 0.$$

Thus, for some $\eta > 0$, estimate (3.9) becomes

$$\begin{aligned}
\mathcal{K}'(t) \geq \eta\Big\{&\mathbb{H}(t) + \|u_t\|_{p+2}^{p+2} + \|\nabla u_t\|_2^2 + \|\nabla u\|_2^2 + (h \circ \nabla u) + \|u\|_{\gamma}^{\gamma}\\
&+ \|\nabla u\|_2^4 + \int\limits_0^1 \|y(x,\rho,t)\|_m^m\, d\rho + \int\limits_{\Omega} |u|^{\gamma}\ln|u|^{k}\, dx\Big\}.
\end{aligned} \tag{3.11}$$

Next, using Holder's and Young's inequalities, we have

$$\left|\int_\Omega u|u_t|^p u_t\,dx\right|^{\frac{1}{1-\alpha}} \le c\Big[\,\|u\|_\gamma^{\frac{\theta}{1-\alpha}} + \|u_t\|_{p+2}^{\frac{\mu}{1-\alpha}}\,\Big], \tag{3.12}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$.

We take $\mu=(p+2)(1-\alpha)$ to get

$$\frac{\theta}{1-\alpha} = \frac{p+2}{(1-\alpha)(p+2)-1} \le \gamma.$$

Further, for $s=\frac{p+2}{(1-\alpha)(p+2)-1}$, estimate (3.12) gives

$$\left|\int_\Omega u|u_t|^p u_t\,dx\right|^{\frac{1}{1-\alpha}} \le c\Big[\,\|u\|_\gamma^{s} + \|u_t\|_{p+2}^{p+2}\Big].$$

Then Lemma 2.3 yields

$$\left|\int_\Omega u|u_t|^p u_t\,dx\right|^{\frac{1}{1-\alpha}} \le c\Big[\,\|u\|_\gamma^{\gamma} + \|u_t\|_{p+2}^{p+2} + \|\nabla u\|_2^2\Big].$$

Similarly, we have

$$\left|\int_\Omega \nabla u\nabla u_t\,dx\right|^{\frac{1}{1-\alpha}} \le c\Big[\,\|\nabla u\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla u_t\|_2^{\frac{\mu}{1-\alpha}}\,\Big],$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$.

We take $\theta=4(1-\alpha)$ to get

$$\frac{\mu}{1-\alpha} = \frac{4}{4(1-\alpha)-1} \le 2,$$

$$\left|\int_\Omega \nabla u\nabla u_t\,dx\right|^{\frac{1}{1-\alpha}} \le c\big\{\|\nabla u\|_2^4 + \|\nabla u_t\|_2^2\big\}.$$

Hence

$$\begin{aligned}
\mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \left(\mathbb{H}^{1-\alpha} + \frac{\varepsilon}{p+1}\int_\Omega u|u_t|^p u_t\,dx + \varepsilon\int_\Omega \nabla u\nabla u_t\,dx + \varepsilon\,\frac{\sigma}{4}\,\|\nabla u\|_2^4\right)^{\frac{1}{1-\alpha}} \\
&\le c\left(\mathbb{H}(t) + \left|\int_\Omega u|u_t|^p u_t\,dx\right|^{\frac{1}{1-\alpha}} + \left|\int_\Omega \nabla u\nabla u_t\,dx\right|^{\frac{1}{1-\alpha}} + \|\nabla u\|_2^{\frac{4}{1-\alpha}}\right) \\
&\le c\Big(\mathbb{H}(t) + \|u\|_\gamma^\gamma + \|u_t\|_{p+2}^{p+2} + \|\nabla u\|_2^2 + \|\nabla u\|_2^4 + \|\nabla u_t\|_2^2\Big) \\
&\le c\bigg(\mathbb{H}(t) + \|u\|_\gamma^\gamma + \|u_t\|_{p+2}^{p+2} + \|\nabla u\|_2^2 + \|\nabla u\|_2^4 + \|\nabla u_t\|_2^2 \\
&\qquad + (h\circ\nabla u) + \int_0^1 \|y(x,\rho,t)\|_m^m\,d\rho + \int_\Omega |u|^\gamma \ln|u|^k\,dx\bigg).
\end{aligned} \tag{3.13}$$

From (3.11) and (3.13), we have

$$\mathcal{K}'(t) \ge \Gamma\mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{3.14}$$

where $\Gamma>0$, this depends only on η and c.

By integration of (3.14), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \Gamma\frac{\alpha}{(1-\alpha)}t}.$$

Hence $\mathcal{K}(t)$ blows-up in time

$$T \leq T^* = \frac{1-\alpha}{\Gamma\alpha\mathcal{K}^{\frac{\alpha}{1-\alpha}}(0)}.$$

□

References

[1] K. Agre and M. A. Rammaha, Systems of nonlinear wave equations with damping and source terms. *Differential Integral Equations* **19** (2006), no. 11, 1235–1270.

[2] M. Al-Gharabli, M. Balegh, B. Feng, Z. Hajjej and S. A. Messaoudi, Existence and general decay of Balakrishnan–Taylor viscoelastic equation with nonlinear frictional damping and logarithmic source term. *Evol. Equ. Control Theory* **11** (2022), no. 4, 1149–1173.

[3] A. V. Balakrishnan and L. W. Taylor, Distributed parameter nonlinear damping models for flight structures. Proceedings Damping. Vol. 89. 1989.

[4] J. M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. *Quart. J. Math. Oxford Ser. (2)* **28** (1977), no. 112, 473–486.

[5] J. D. Barrow and P. Parsons, Inflationary models with logarithmic potentials. *Phys. Rev. D* **52** (1995), 5576–5587.

[6] K. Bartkowski and P. Górka, One-dimensional Klein–Gordon equation with logarithmic nonlinearities. *J. Phys. A* **41** (2008), no. 35, 355201, 11 pp.

[7] R. W. Bass and D. Zes. Spillover, nonlinearity, and flexible structures. *NASA. Langley Research Center, Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems*, Part 1. 1991.

[8] I. Białynicki-Birula and J. Mycielski, Wave equations with logarithmic nonlinearities. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **23** (1975), no. 4, 461–466.

[9] D. R. Bland, *The Theory of Linear Viscoelasticity.* Courier Dover Publications, 2016.

[10] S. Boulaaras, A. Choucha, D. Ouchenane and B. Cherif, Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms. *Adv. Difference Equ.* **2020**, Paper no. 310, 10 pp.

[11] S. Boulaaras, A. Draifia and Kh. Zennir, General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan–Taylor damping and logarithmic nonlinearity. *Math. Methods Appl. Sci.* **42** (2019), no. 14, 4795–4814.

[12] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. Ferreira, Existence and uniform decay for a non-linear viscoelastic equation with strong damping. *Math. Methods Appl. Sci.* **24** (2001), no. 14, 1043–1053.

[13] H. Chen, P. Luo and G. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. *J. Math. Anal. Appl.* **422** (2015), no. 1, 84–98.

[14] A. Choucha and S. Boulaaras, Asymptotic behavior for a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan–Taylor damping. *Bound. Value Probl.* **2021**, Paper no. 77, 16 pp.

[15] A. Choucha, S. Boulaaras, D. Ouchenane and S. Beloul, General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan–Taylor damping, logarithmic nonlinearity and distributed delay terms. *Math. Methods Appl. Sci.* **44** (2021), no. 7, 5436–5457.

[16] A. Choucha, S. Boulaaras, D. Ouchenane, B. B. Cherif and M. Abdalla, Exponential stability of swelling porous elastic with a viscoelastic damping and distributed delay term. *J. Funct. Spaces* **2021**, Art. ID 5581634, 8 pp.

[17] A. Choucha, D. Ouchenane and S. Boulaaras, Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term. *Math. Methods Appl. Sci.* **43** (2020), no. 17, 9983–10004.

[18] A. Choucha, D. Ouchenane and S. Boulaaras, Blow-up of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms. *J. Nonlinear Funct. Anal.* **2020**, Article ID 31, 10 pp.

[19] A. Choucha, D. Ouchenane, Kh. Zennir and B. Feng, Global well-posedness and exponential stability results of a class of Bresse–Timoshenko-type systems with distributed delay term. *Math. Methods Appl. Sci.* **2020**, 26 pp.; DOI: `https://doi.org/10.1002/mma.6437`.

[20] B. D. Coleman and W. Noll, Foundations of linear viscoelasticity. *Rev. Modern Phys.* **33** (1961), 239–249.

[21] B. Feng and Y. H. Kang, Decay rates for a viscoelastic wave equation with Balakrishnan–Taylor and frictional dampings. *Topol. Methods Nonlinear Anal.* **54** (2019), no. 1, 321–343.

[22] B. Feng and A. Soufyane, Existence and decay rates for a coupled Balakrishnan–Taylor viscoelastic system with dynamic boundary conditions. *Math. Methods Appl. Sci.* **43** (2020), no. 6, 3375–3391.

[23] B. Gheraibia and N. Boumaza, General decay result of solutions for viscoelastic wave equation with Balakrishnan–Taylor damping and a delay term. *Z. Angew. Math. Phys.* **71** (2020), no. 6, Paper no. 198, 12 pp.

[24] P. Górka, Logarithmic Klein–Gordon equation. *Acta Phys. Polon. B* **40** (2009), no. 1, 59–66.

[25] L. Gross, Logarithmic Sobolev inequalities. *Amer. J. Math.* **97** (1975), no. 4, 1061–1083.

[26] L. Guo, Z. Yuan and G. Lin, Blow up and global existence for a nonlinear viscoelastic wave equation with strong damping and nonlinear damping and source terms. *Appl. Math., Irvine* **06** (2015), no. 05, Article ID: 56316, 10 pp.

[27] M. Kafini and S. Messaoudi, Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay. *Appl. Anal.* **99** (2020), no. 3, 530–547.

[28] G. Kirchhoff, *Vorlesungen über Mechanik.* Tauber, Leipzig, 1883.

[29] W. Liu, B. Zhu, G. Li and D. Wang, General decay for a viscoelastic Kirchhoff equation with Balakrishnan–Taylor damping, dynamic boundary conditions and a time-varying delay term. *Evol. Equ. Control Theory* **6** (2017,) no. 2, 239–260.

[30] F. Mesloub and S. Boulaaras, General decay for a viscoelastic problem with not necessarily decreasing kernel. *J. Appl. Math. Comput.* **58** (2018), no. 1-2, 647–665.

[31] C. Mu and J. Ma, On a system of nonlinear wave equations with Balakrishnan–Taylor damping. *Z. Angew. Math. Phys.* 65 (2014), no. 1, 91–113.

[32] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Optim.* **45** (2006), no. 5, 1561–1585.

[33] E. Pişkin, J. Ferreira, H. Yuksekkaya and M. Shahrouzi, Existence and asymptotic behavior for a logarithmic viscoelastic plate equation with distributed delay. *Int. J. Nonlinear Anal. Appl.* **13** (2022) no. 2, 763–788.

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